Department of Aeronautics and Astronautics

College of Engineering

UNIVERSITY OF WASHINGTON

Photothermoviscoelasticity II

Status Report

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by

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PHOTOTHERMOVISCOELASTIC RESEARCH

by

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A. SUMMARY

This report presents the status of the research project being conducted in the Department of Aeronautics and Astronautics at the University of Washington under Research Grant NsG-401 from the National Aeronautics and Space Administration. This report covers work completed during the period October 1, 1965 through April 1, 1966. It is for use of the technical monitor of the grant and is not intended for publication or general distribution.

The project consists of an analytical and experimental study to establish a procedure for stress analysis of a viscoelastic structure subject to transient temperature and time-dependent loadings. The extension of photoelasticity to this problem has been described in previous reports. The analytical basis for the interpretation of photoviscoelastic observations in the linear range, details of the experimental procedure, and example problems were presented in September 1965.* That report is in preparation by NASA as a formal NASA publication. This document describes subsequent work to further extend the method and to refine the established procedures.

Section B contains a description of further work on the linear photoviscoelastic problems. Modifications to the photoelastic

*"Photoviscoelasticity" by E. H. Dill and C. W. Fowlkes, University of Washington, Department of Aeronautics and Astronautics, Report 65-1.

bench are described. The direct reduction of data from a viscoelastic test by use of a uniaxial test specimen as an analog computer is presented. Several problems which illustrate photoviscoelastic procedures and which are of practical interest are solved.

The extension of the embedded polariscope method to low modulus material is also described in Section B. This technique is useful either for elastic or viscoelastic materials where the "film" of polaroid would substantially alter the stress distribution. Three-dimensional models with transient stress distribution can be studied by this means.

The solution of a viscoelastic problem with non-uniform temperature distribution is given in Section C. We are now in the process of obtaining an analytical solution of this problem to compare with the experimental results.

In order to extend the photoviscoelastic technique to dynamic loading, we require a means of rapidly rotating the plane of polarization of the analyzer and polarizer. An invention is described in Section D in which the plane of polarization of a polarizer is rotated electronically.

The fundamental theory for the study of non-linear viscoelasticity and elasticity problems is presented in Section E. The nature of constitutive relations for the mechanical and electromagnetic behavior of high polymers experiencing large strains is derived. The use of such relations for a given material to solve stress analysis problems for non-linear viscoelasticity and non-linear elasticity by an extension of photoelasticity is indicated.

B. EXPERIMENTAL METHODS

by

C. W. Fowlkes

1. INTRODUCTION

In this section, the progress on the experimental techniques for linear photothermoviscoelasticity is reviewed. Improvements in the equipment are described along with the presentation of new experimental techniques. The results of several new tests are presented.

Included in this section is a description of the analog method of data reduction which considerably simplifies the determination of the stress history in the model. A paper describing the analog method has been accepted for publication by the Society for Experimental Stress Analysis. Several experiments are described in this report in which the analog method was employed for data reduction.

Preliminary experiments have been performed on the development of an embedded polariscope technique to be used in three-dimensional viscoelastic models. The results of these experiments are quite promising and several aspects of the technique have been established. An actual three-dimensional test will be performed in the future.

2. PHOTOVISCOELASTIC BENCH

Improvements have been made on the rotating polaroid bench: a 35mm pulse-cine camera, Northridge Research Type RF 12, has been added to replace the 16mm Cine Special. The 35mm camera is triggered by an electrical pulse for single frame operation at rates

to 12 frames/second and will operate cine at 20 frames/second. A pulse generator designed and built in our laboratory produces shaped pulses at frequencies continuously adjustable between 5 pulses/second and 1 pulse each 120 seconds. This pulse generator switches 110 vdc into the polaroid stepping solenoids and, through a time delay, switches 28 vdc into the 35mm camera. Thus, for every pulse into the system, the polaroids rotate a fixed interval, stop, and then a single picture is taken. The image of a clock contained within the camera is included on each frame.

The increased resolution of the larger 35mm negative improves the accuracy of determining fringe orders in regions of stress concentration. The semi-automatic operation of the new system simplifies the testing procedure. A photograph of the rotating polaroid bench with the 35mm camera is shown in Figure 1.

3. ANALOG METHOD OF DATA REDUCTION

In general, there can be time variations of the isochromatic fringe order, the isoclinic orientation, and the temperature at each point in a viscoelastic model. A procedure for determining the history of the principal stress difference at any point in the model was discussed in previous reports. That procedure can be applied if the material is linear viscoelastic and optically linear. A tensile calibration was performed to determine the mechanical and optical properties of the model material over the range of stresses and temperatures existing in the model and the observations from an experiment were processed numerically using the calibration data.

Another approach for determining the history of difference of principal stresses from observed fringe order is the following analog method. A simple model, such as a tensile specimen, whose state of stress is known is constructed. This model is then subjected to the same temperature history as a particular point of the prototype while the loads are adjusted to produce the same isochromatic and isoclinic history as observed in the prototype. The principal stress difference history is then computed for the model from the boundary loads. The result is the principal stress difference history for the prototype.

In principle, the analog method could be used to reduce the data from any test. In practice, however, accounting for the isoclinic variation in an analog model is difficult. The analog method is most useful when the isoclinic orientation is constant. For such problems, the analog model can be a simple tensile specimen. Several problems of this type are analyzed below.

The analog solutions were performed on the tensile creep apparatus pictures in Figure 2. The variation in fringe order was sensed by a photomultiplier tube, the successive maximums and minimums being successive half fringe orders. The fringe order history at the point of the complex model to be analyzed was transferred to the chart of one channel of a brush recorder in the form of maximums and minimums of light intensity versus time. Partial fringe orders were plotted between the maximums and minimums and a continuous sinusoidal master curve of light intensity was drawn through the points. The model was then loaded in such a manner as to match

the output of the photomultiplier tube to this master curve. This load was recorded simultaneously from a strain gage load cell.

The required load rate was relatively low and loading could be done by hand. Several tests show that short time amplitude errors of ±20% in plotting and tracking the master fringe order curve produce variations of only about ±5% in the final stress history. After a few practice runs one could follow the master curve quite easily and get results reproducible to within 6%. An electronic servo would be helpful if a very large number of solutions were to be run.

4. MODEL MATERIAL

The model material was a mixture of epoxy resins supplied by CIBA Products Company, Fair Lawn, New Jersey. A mixture of Araldite 502 casting resin and Araldite 508 flexible modifier with triethylenetetramine hardener can be prepared having a range of relaxation times suitable for photoviscoelastic model experiments. The materials are mechanically and optically linear for a wide range of stresses and temperatures, and have less than 1% permanent creep. The materials used for models reported here were mixes of Araldite 502 and 508 with 10% hardener. The resin components are heated to 125°F., mixed thoroughly, and cast into sheets between glass plates. Wiping the glass plates with commercial Hysol AC4-4368 mold release agent will prevent adhesion to the mold. The mixture is cured for 12 hours at 160°F.

5. VISCOELASTIC PLATE WITH HOLE

A strip three inches wide and eight inches long was cut from a cast sheet of ¼-inch thick material and a ½-inch diameter hole machined in its center. Rigid grips were bonded along the shorter sides. A tensile creep specimen was prepared from the same sheet and tested (Figure 3).

The model was placed in a light field on the photoviscoelastic bench described in Section 2 and shown in Figure 1. In one test plane, polarizing sheets (Polaroid HN32) were used and rotated during the test to monitor the isoclinic patterns. In a second test, fixed polarizing sheets having attached quarter wave plates were used to check the isochromatic patterns. A weight was hung on the specimen and the specimen was allowed to creep under the constant load.

Photographs were made of the changing fringe patterns and the data was subsequently read from the film using an enlarger and a microscope. A print of the model fringe pattern at 10,500 seconds after loading is reproduced in Figure 4. Zero time corresponds to the application of the load. The isoclinic patterns did not vary during the test.

The isochromatic history at the point of maximum stress at the edge of the hole was read from the film. This curve differs only by a constant from the curve representing the isochromatic history in a tensile creep (constant stress) test (Figure 5). This constant is the ratio of stresses. The stress at the edge of the hole is thus found to be constant with time and its magnitude

is easily computed using the tensile calibration. This test is trivial in that the outcome can be predicted knowing that the material is linear viscoelastic and having a solution for the stresses in an elastic material in this configuration.

A Hysol 4485 (elastic) model of the same dimensions was prepared and tested to determine the stress at the edge of the hole. These results agreed with the photoviscoelastic test and the analytical solution.

6. VISCOELASTIC PLATE WITH UNBONDED RIGID INCLUSION

The hole of the viscoelastic plate (Section 5) was then fitted with an aluminum disc. The aluminum disc was machined so that it would slide smoothly into the hole of the relaxed viscoelastic plate after a layer of Teflon tape had been bonded around its edge. The model was placed in the polarized light field and loaded in tension with a weight. Selected photographs taken during the test are shown in Figure 6. Comparing the fringe patterns from the two tests reveals a difference as the strains become large. In this test, as the model strains the sides of the hole bear against the rigid disc while the top of the hole pulls away from the surface of the disc.

The maximum shear stress still occurs at the side of the hole where it bears against the disc. The fringe order history at this point was read and was found to be nearly identical with the fringe order history in the plate without the inclusion. The isoclinic at this point was constant during the test.

A Hysol 4485 model was fitted with a solid disc and the maximum fringe order at the edge of the hole was compared to the fringe order in the same plate without the disc. No difference in the fringe order versus applied stress behavior could be found.

7. VISCOELASTIC PLATE WITH BONDED RIGID INCLUSION

A viscoelastic plate 8 inches long, 3 inches wide, and 0.25 inches thick was prepared with a ½-inch diameter hole. An aluminum disc was machined to fit into the relaxed plate with approximately 0.002 diametrical clearance. The disc was bonded into the plate. Adhesive used for bonding the rigid disc into the plate consisted of an identical mix of the epoxy used to case the plate itself.

The plate was placed in the polariscope and loaded with a weight. The fringes were recorded on the 35mm pulse camera. One run was made with plane polarizers rotating in a stepwise manner for recording the isoclinic history. During the initial moments of the test the isoclinics obscured the rapidly changing isochromatics so a second run was made using circularly polarized light. A more accurate observation of the isochromatics could be made in this second run. Figure 7 shows the isoclinics at 70 seconds after loading; this is typical of the pattern throughout the test. The isochromatics are shown in Figure 8. These photographs were selected from those made during the second test. The maximum fringe order in this test occurred at points on the centerline of the model 0.125 inches above and below the rigid disc.

The fringe order history at these points is shown in Figure 9. The isoclinic orientation at these points was constant.

In a plate with a hole, points of high stress occur at the sides and the top and bottom surfaces of the hole. These same points exhibited nearly zero fringe order in the plate with the bonded rigid disc. A point of high stress in the plate with the rigid disc occurred along the sides of the disc 0.125 inches away from the disc. The fringe order history here lagged at the maximum point by approximately ½ throughout the test. An analog solution was run for this point. The principal stress difference history as determined from the analog is shown in Figure 10.

8. VISCOELASTIC PLATE WITH BONDED ELASTIC INCLUSION

A viscoelastic plate of the same dimensions as the preceeding tests was prepared. A ½-inch diameter circular disc of low modulus (465 psi) Hysol 4485 was machined to a loose fit with the hole in the viscoelastic plate. As in the test of Section 7, the disc was bonded into the plate with the same epoxy as was used to cast the plate. It is important to match the properties of the adhesive to the properties of the plate. Even a thin layer of rigid adhesive would have altered the fringe pattern considerably.

The plate was placed in the polariscope and loaded with a weight. Tests were run with plane polarized light and with circularly polarized light as in Test 7. The isoclinics at 2,000 seconds after loading are shown in Figure 11. This pattern is fairly typical of the isoclinic pattern throughout the test. Selected

photographs of the isochromatics occurring in circularly polarized light are shown in Figure 12.

The isochromatic patterns of this test show some similarities to the previous test with the rigid disc. The fringe order at the sides of the hole and at the top and bottom of the hole remained less than 1 throughout the test. The maximum fringe order again occurred 0.125 inch above and below the inclusion and is shown in Figure 13. This same point is the point of minimum fringe order in a simple plate with a hole. An analog solution was run and the history of the maximum principal stress difference was determined (Figure 14).

9. NON-HOMOGENEOUS VISCOELASTIC PLATE

A loading jig was constructed to apply uniform pressure along the edge of a low modulus plate (Figure 15). Air pressure is applied through a thin latex diaphragm. Plexiglas plates contain the model and prevent it from buckling. A copious layer of silicone oil is spread between the model and the plexiglas, keeping friction to an insignificant level. The whole jig can be placed in the polariscope since the loads on the plexiglas are well below the level necessary to cause any fringes. Pressure is supplied to the model by filling a storage tank to the predetermined pressure and then opening a valve. The volume of the jig is very small compared to the volume of the storage tank.

A diagram of the test specimen is shown in Figure 16. A homogeneous calibration specimen was made of Hysol 4485 to deter-

mine the actual pressure acting on the model for a given air pressure. Photographs of the loaded Hysol model are shown in Figure 17 and the calibration curve in Figure 18. The photographs were taken several weeks after the calibration and show a considerable edge fringe. This disturbance was not evident when the actual calibration was performed.

Two sheets, A and B, having different relaxation times were cast in the laboratory. The relaxation time is varied by changing the proportions when mixing the epoxy resins. Sheet A was a 50/50 mix of Araldite 502 and 508 plus 10% hardener, and Sheet B was a 45/50 mix plus 10% hardener. The tensile creep compliances of A and B are shown in Figure 19. Blocks of these materials 2-5/8 inches by 3 inches were machined and the 2-5/8 inches edges of one "A" block and one "B" block were bonded together. The epoxy mixture used to cast Sheet A was used for the adhesive to join Blocks A and B, thus insuring that no further non-homogeneity would be introduced. One 6 inch edge of the composite model was then bonded securely to a ¼-inch square aluminum bar. The opposite 6 inch edge which was to be exposed to the pressure loading was then machined to the required final dimension. A high speed router was used to produce a smooth edge.

The model was allowed to relax at testing temperature (70°F) for 12 hours. It was then wiped with a generous layer of low viscosity silicone oil and placed in the calibrated loading jig. The model was loaded with a step input of 15 psi air pressure (11.1 psi effective pressure) and the resulting fringe patterns

were photographed with the 35mm pulse camera.

The model was tested first in plane polarized light with rotating polaroids to determine the isoclinic history. A set of the photographs taken 5 seconds, 30 seconds, and 10,000 seconds after loading are shown in Figure 20. Another test was performed with circularly polarized light to check the isochromatics. Photographs selected at several times after loading are shown in Figure 21. The fringe order data was read from the film using an enlarger with a microscope being used to view high fringe order regions of the negative.

The maximum fringe order in both A and B occurred on the pressurized surface near the A-B interface. As near as could be determined, the isoclinic orientation at these points was constant throughout the test (Figure 19). The isochromatic histories for A and B are shown in Figure 22.

Analog solutions were performed to determine the history of the principal stress difference for these points in part A and B of the model. These solutions are shown in Figure 23, along with the solution for a homogeneous model. The stress in the non-homogeneous model is temporarily increased during the early part of the test and then relaxes toward homogeneous model value for times of the order of the relaxation time for the material. Enlargements of the model show the irregularity in the loaded boundary as the lower modulus (B) part experiences a larger strain than the higher modulus (A) part. This irregularity becomes less pronounced near the end of the test.

10. EMBEDDED POLARISCOPE

The embedded polaroid technique described here was developed for studying the stresses in three-dimensional models of low modulus material. One possible application is gravity loading of low modulus materials. For these experiments, the model material must have a very low modulus in order to maintain similarity when loaded by gravity. The ordinary embedded polariscope is much too stiff for this application. Another possible application is the study of three-dimensional viscoelastic models in which the changing fringe patterns must be observed so that frozen stress techniques cannot be used.

The standard embedded polaroid technique consists of bonding a polarizing sheet to a thin slice which is cast into a three-dimensional model. The model is placed in a tank and the fringe patterns in the embedded sheet are observed. This technique has been used in high modulus elastic models, however, special problems arise in using this technique for low modulus elastic and viscoelastic models.

Available polarizing films have a modulus of about 10^5 psi. The model materials of interest have moduli in the region of 10^2 psi to 10^3 psi. The polarizing film thus cannot be placed directly into these models without drastically altering the stress field when the model is strained. However, the reinforcing effect of the polarizing film can be considerably reduced if it is divided into very small pieces with spaces between the pieces. Two techniques for cutting and applying the polarizing film will be de-

scribed and the performance of a model having a finely divided polarizing film bonded to it will be presented below.

10.1. POLAROID PREPARATION

A polarizing film incorporating ¼ λ compensation (Polaroid HNCP 37) was used for these experiments. The polarizing sheet, as received from the manufacturer, consists of a polarizing film 0.0025 inches thick sandwiched between transparent sheets of plexiglas. If the polarizing sheet is immersed in dichloromethane, the plexiglas will soften and may be scraped off the polarizing film. Two or three immersions may be required to remove the plexiglas completely. The film may tend to curl and is best stored under a weighted flat surface.

METHOD I.

The film is sliced in the jig shown in Figure 24. A sheet of 0.010 latex rubber (dental dam) is bonded to the platen and the polarizing film is bonded to the rubber with contact cement. The slicing head has four razor blade knives clamped between ball bearing rollers. The rollers serve to hold the film as the knives slice through the film. The slicing head is guided by the indexing T-square of the slicing jig. The film can thus be sliced into uniform ribbons. The platen is rotated 90° and the process repeated, leaving the polarizing film sliced into uniform squares.

The rubber sheet is then peeled from the platen. The rubber sheet is placed on a flat surface and uniformly stretched to get

the desired spacing between the squares. The area around the patch of polaroid squares is coated with an appropriate release agent (polyvinyl alcohol or Hysol AC4-4368 mold release agent).

The squares are bonded to the sheet of model material using an uncured mixture of the model material itself as the adhesive. The use of a high modulus adhesive would interfere considerably with the stress distribution in the model. The area to be bonded is coated with adhesive and placed against the polaroid patch and allowed to cure. After cure, the rubber sheet may be peeled off leaving the squares on the model.

Polaroid film may be bonded to the opposite side of the model slice in the same manner effecting a transmission polariscope. An alternative procedure is to coat the opposite side of the slice with a reflecting material. This procedure has the advantage of doubling the number of isochromatic fringes observed. A film of aluminum deposited by vacuum has been successfully used for a reflector. If the film is very thin, many small cracks will form when it is loaded, destroying its reinforcing effect.

This slicing method has the advantage of being able to adjust the spacing between the squares of polaroid. The several operations involved, however, take time and technique. Method II has proven to be a more reliable approach for most tests.

METHOD II.

A polarizing film is prepared and bonded directly to the model with an adhesive consisting of the mixture used to cast the model

material itself. The use of such an adhesive is essential to insure homogeneity of the final model. Care must be taken to insure that the film is bonded smoothly to the model. The adhesive mixture should be deaerated in a vacuum and the film applied to the model and rolled to force out trapped air. The adhesive is first allowed to cure at a temperature slightly above room temperature and then cured at its normal schedule. If the model is placed directly in a hot oven, small bubbles will form between the polarizing film and the model.

The polarizing film is cut into squares with the apparatus shown in Figure 25. The cutting head consists of a gang of ten slitting saws 0.2 inches apart and each 0.006 inches thick mounted on a mandrel in a vertical mill. The model is fastened to a flat angle bracket with double coated tape. The saws are set deep enough to cut through the film and 0.001 to 0.003 inches into the model. A mandrel speed of 2700 rpm and a feed of 5 inches/minute set to climb mill works very well. The slots cut into the sheet do not interfere with the model behavior as they are subsequently filled with model material when the sheet is cast into the final three-dimensional model.

10.2 CALIBRATION

A calibration test was performed on a Hysol 4485 plate with a hole (Figure 26). One part of the model was covered with bonded polaroid squares and a symmetrical part was left undisturbed for viewing in conventional polarized light. The model was loaded in

tension and the maximum fringe order at the edges of the hole was observed. The fringe order at the edges of the hole was observed. The fringe order was the same in the bonded polaroid region as it was in the undisturbed region, thus illustrating that the effect of the bonded polaroid squares is sufficiently small.

Figure 27 shows a cylinder of viscoelastic material in a Plexiglas case which has been instrumented with sheets having polaroid squares bonded to both sides.

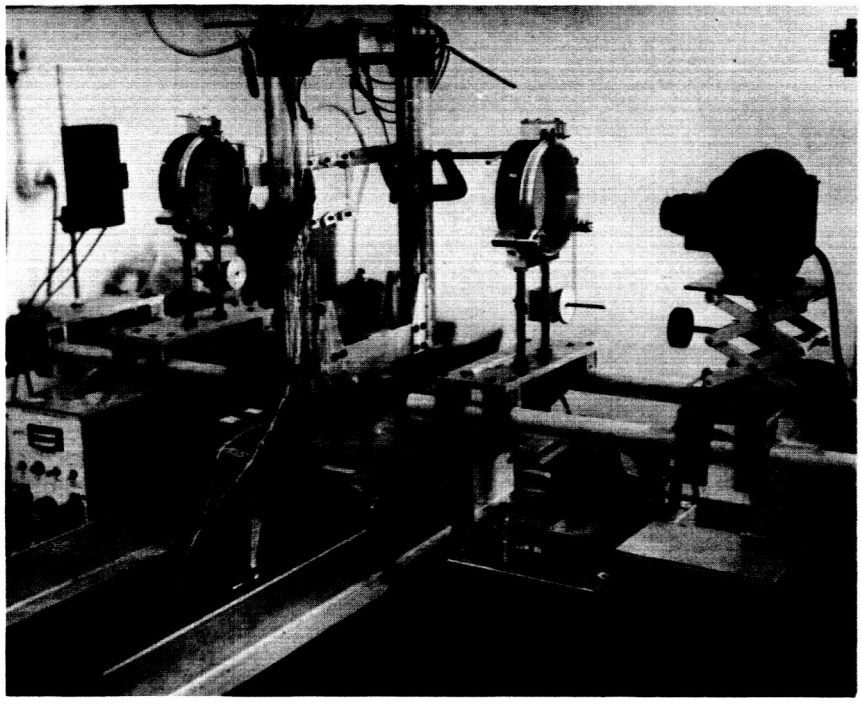


Figure 1. Photoviscoelastic Bench

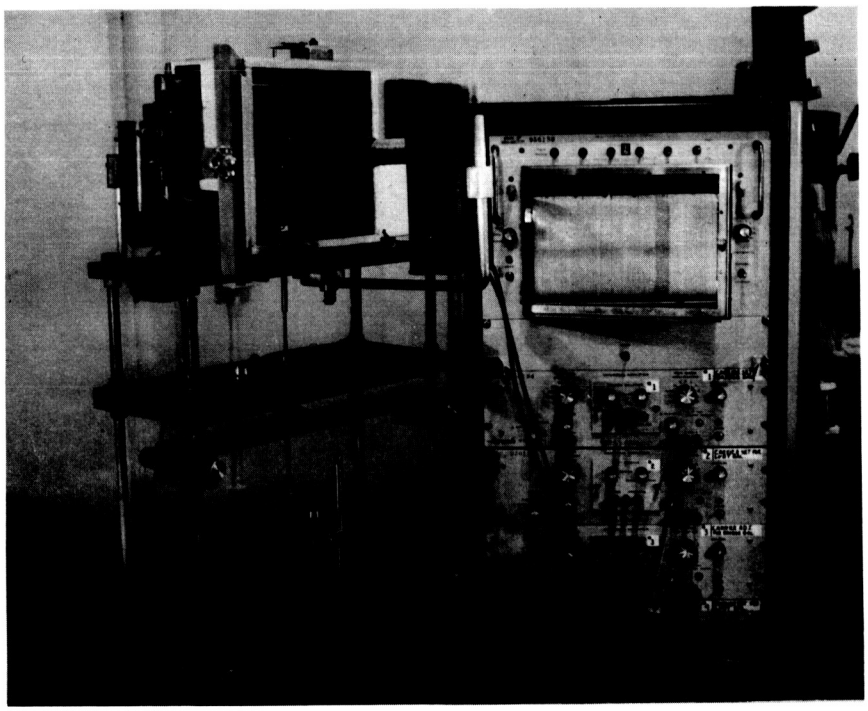


Figure 2. Tensile Creep Apparatus

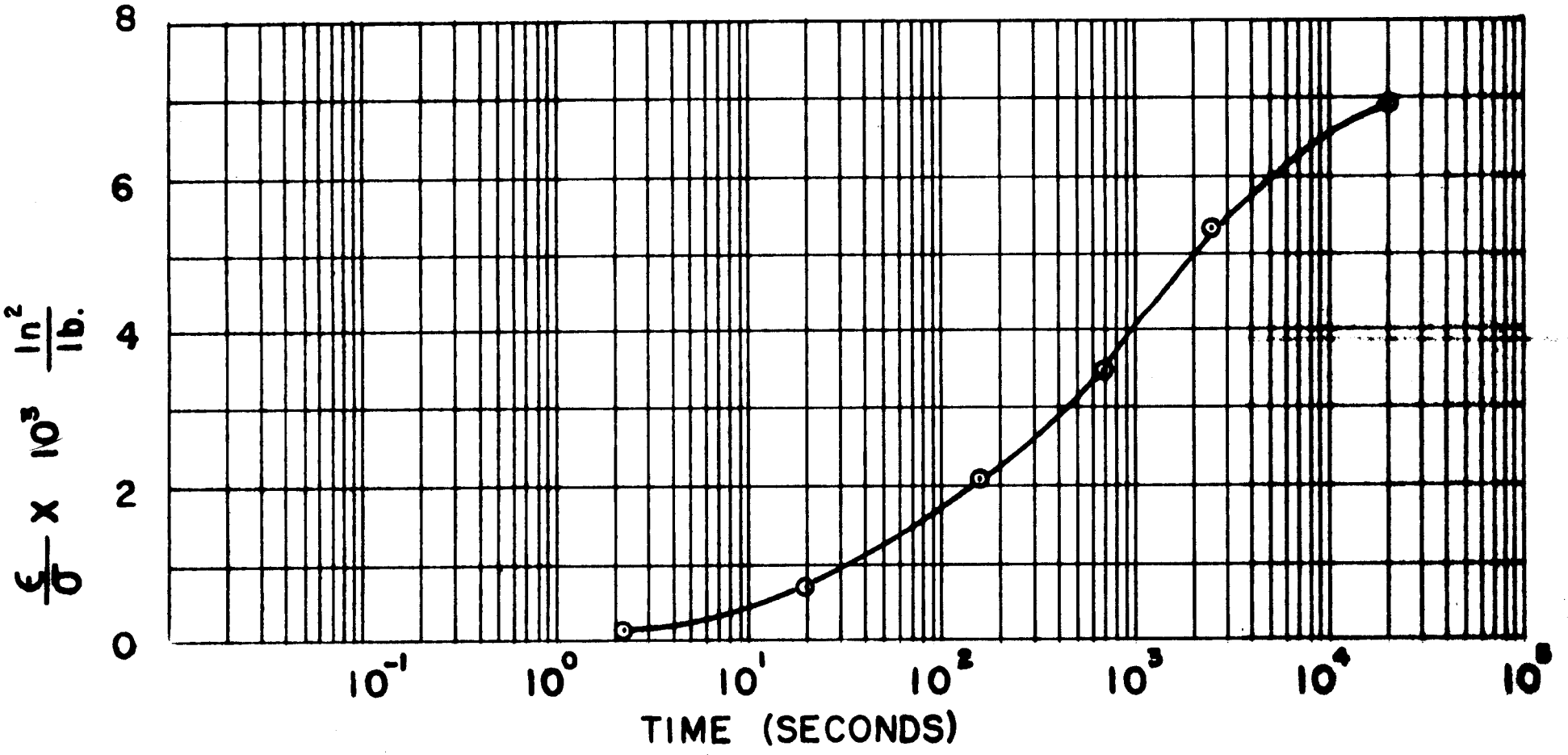
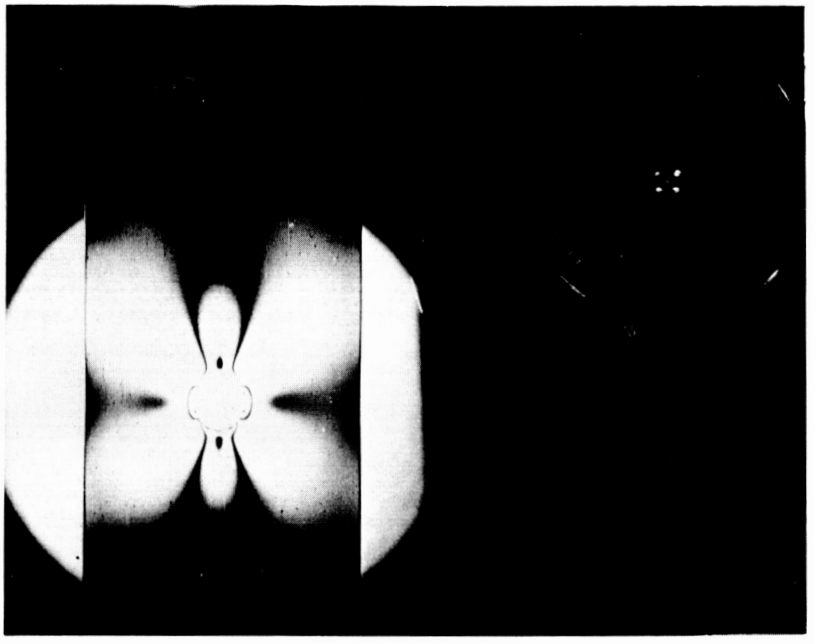


Figure 3. Creep Compliance.



10,500 sec.

Figure 4. Plate with Hole, Light Field.

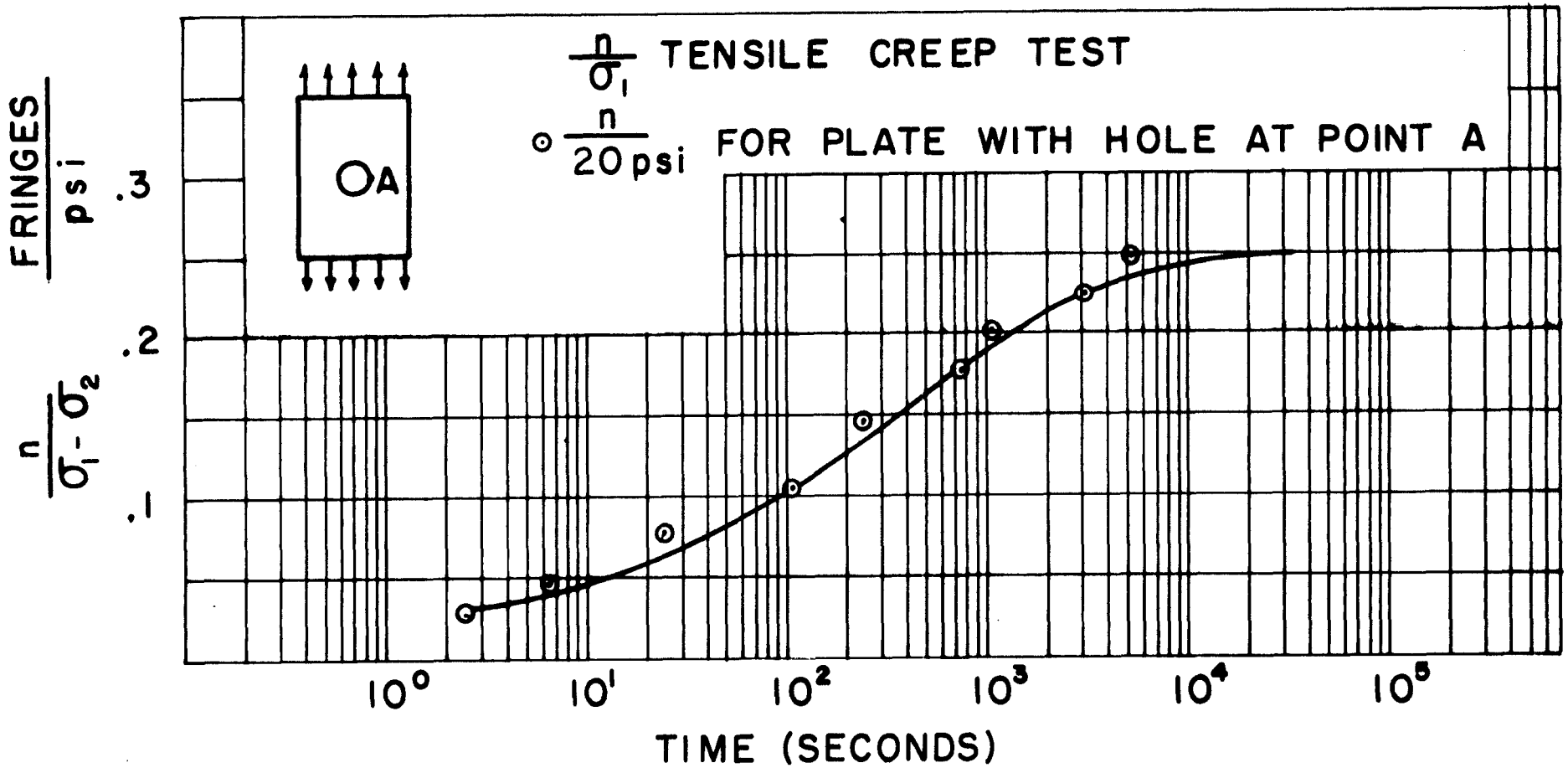
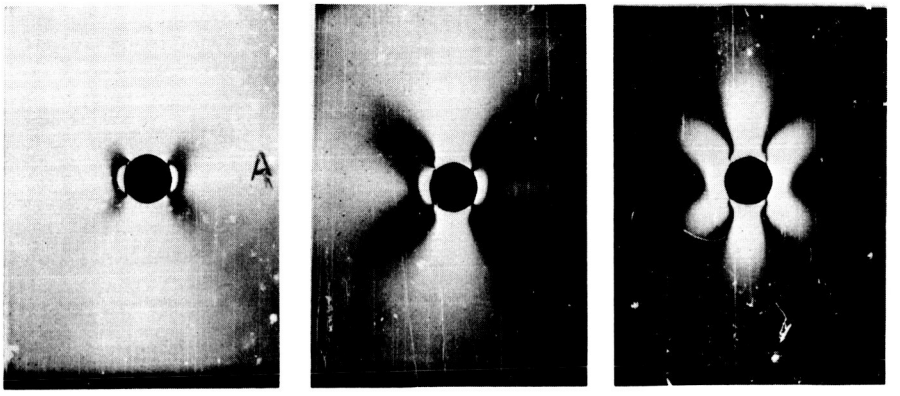
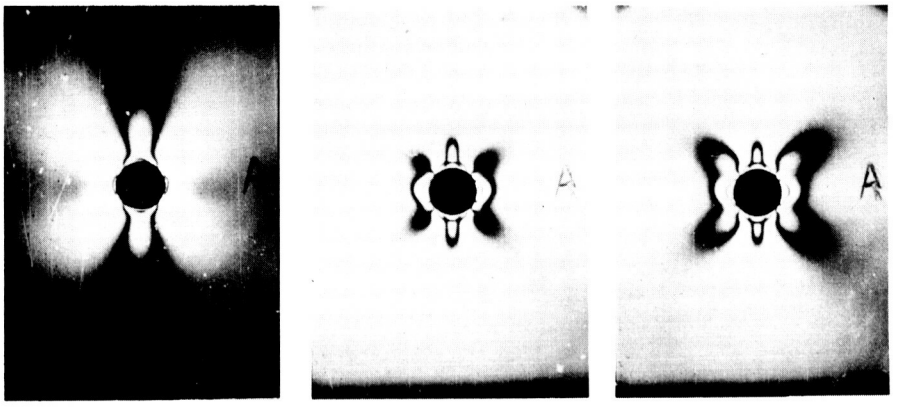


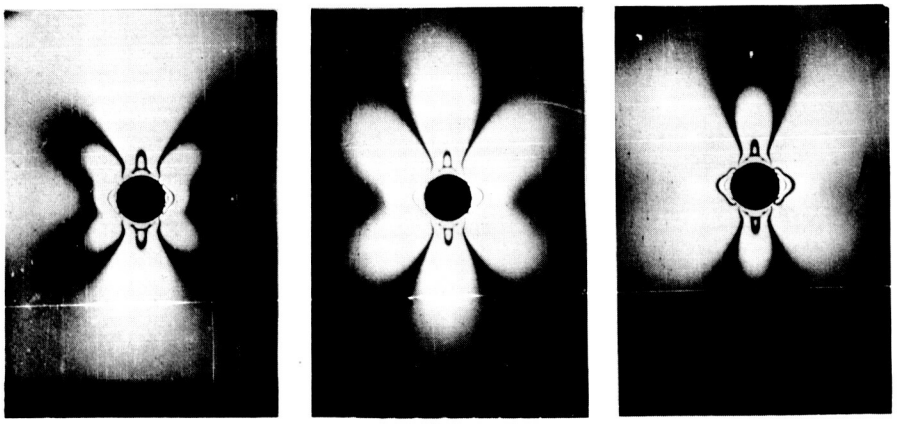
Figure 5. Optical Creep Compliance.



12 sec. 28 sec. 48 sec.

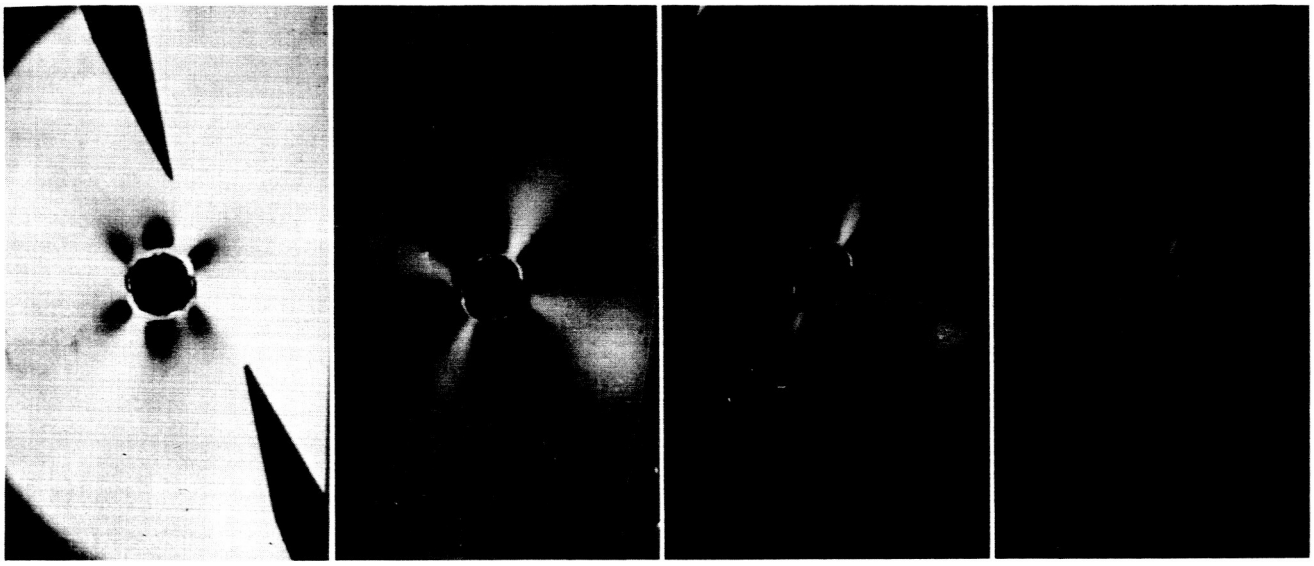


100 sec. 870 sec. 1260 sec.

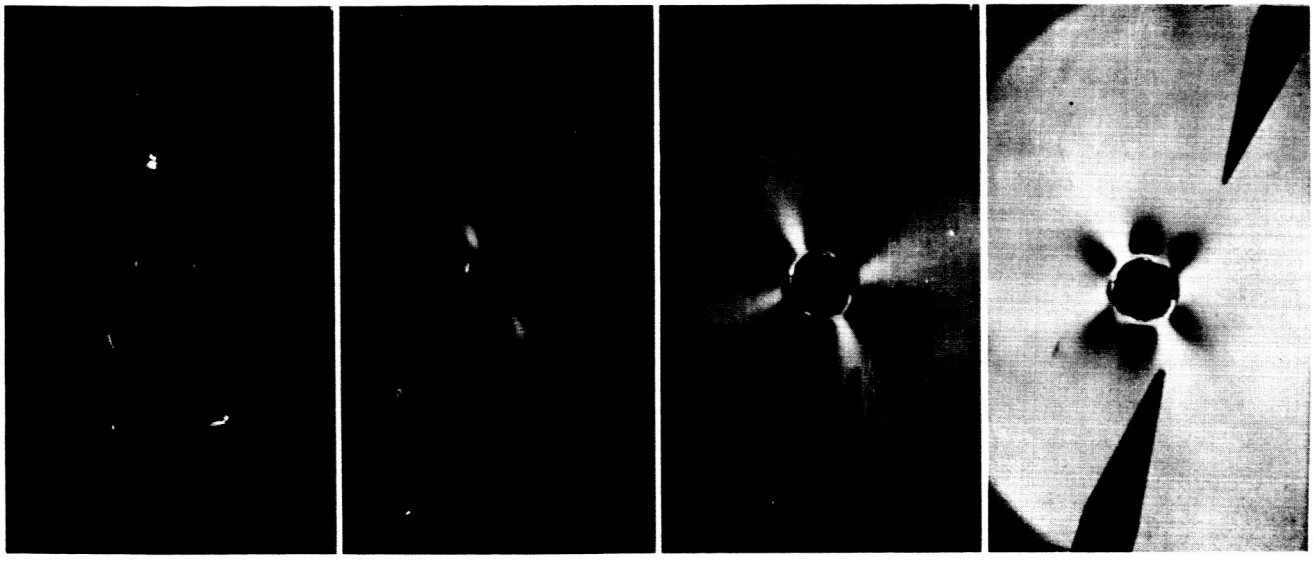


2160 sec. 4320 sec. 12,300 sec.

Figure 6. Plate with Unbonded Rigid Inclusion, Light Field.



$\alpha = -20^\circ$ $\alpha = -10^\circ$ $\alpha = -8^\circ$ $\alpha = -4^\circ$



$\alpha = 0^\circ$ $\alpha = 4^\circ$ $\alpha = 10^\circ$ $\alpha = 20^\circ$

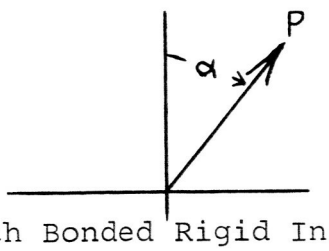
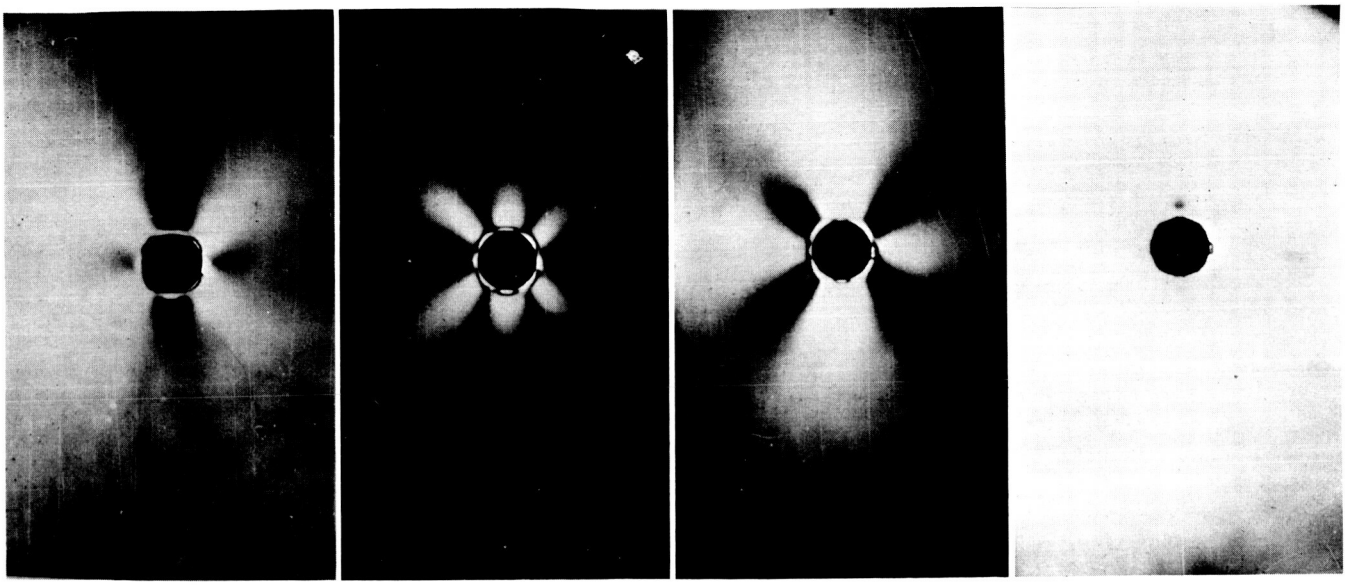
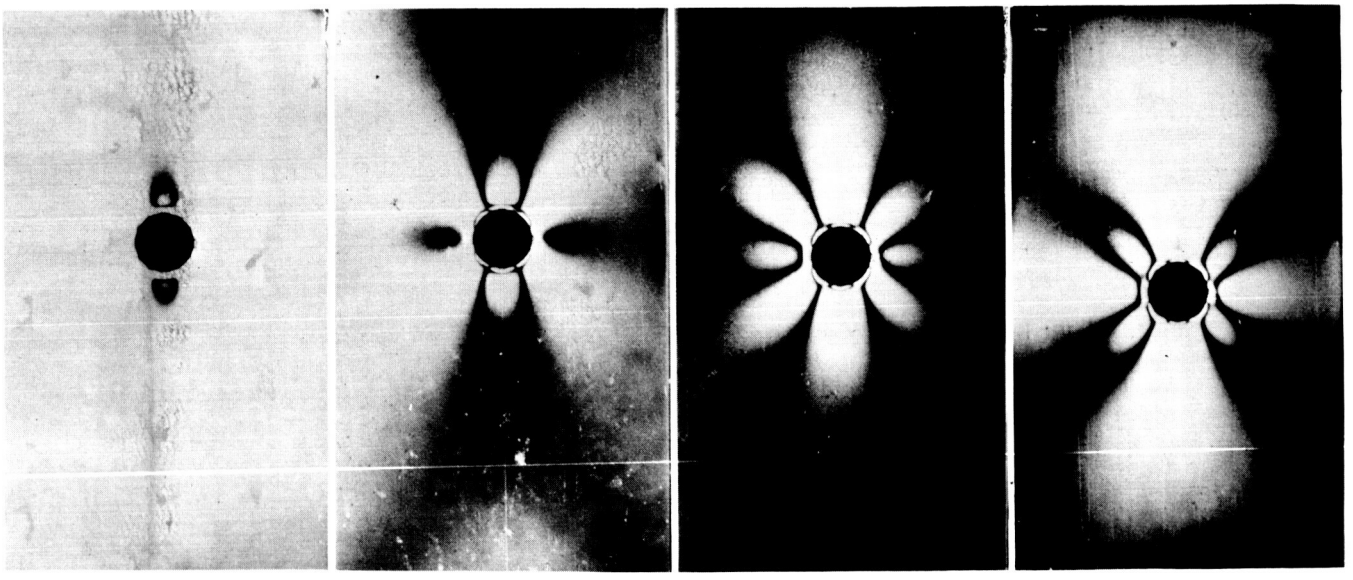


Figure 7. Plate with Bonded Rigid Inclusion, Isoclinics.



20 sec. 50 sec. 100 sec. 300 sec.



700 sec. 2000 sec. 5100 sec. 18,500 sec.

Figure 8. Plate with Bonded Rigid Inclusion, Isochromatics, Light Field.

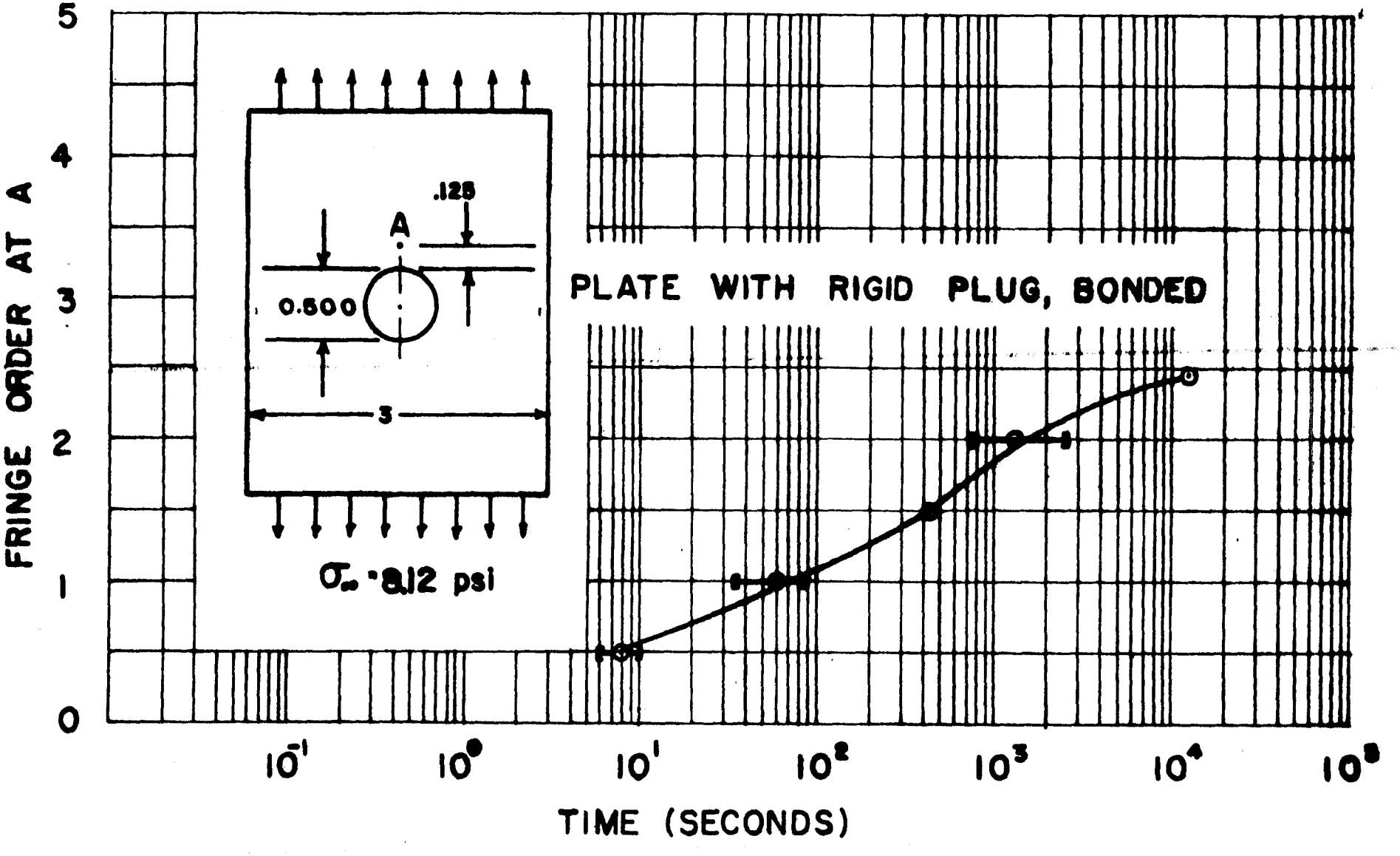


Figure 9. Fringe Order History at A.

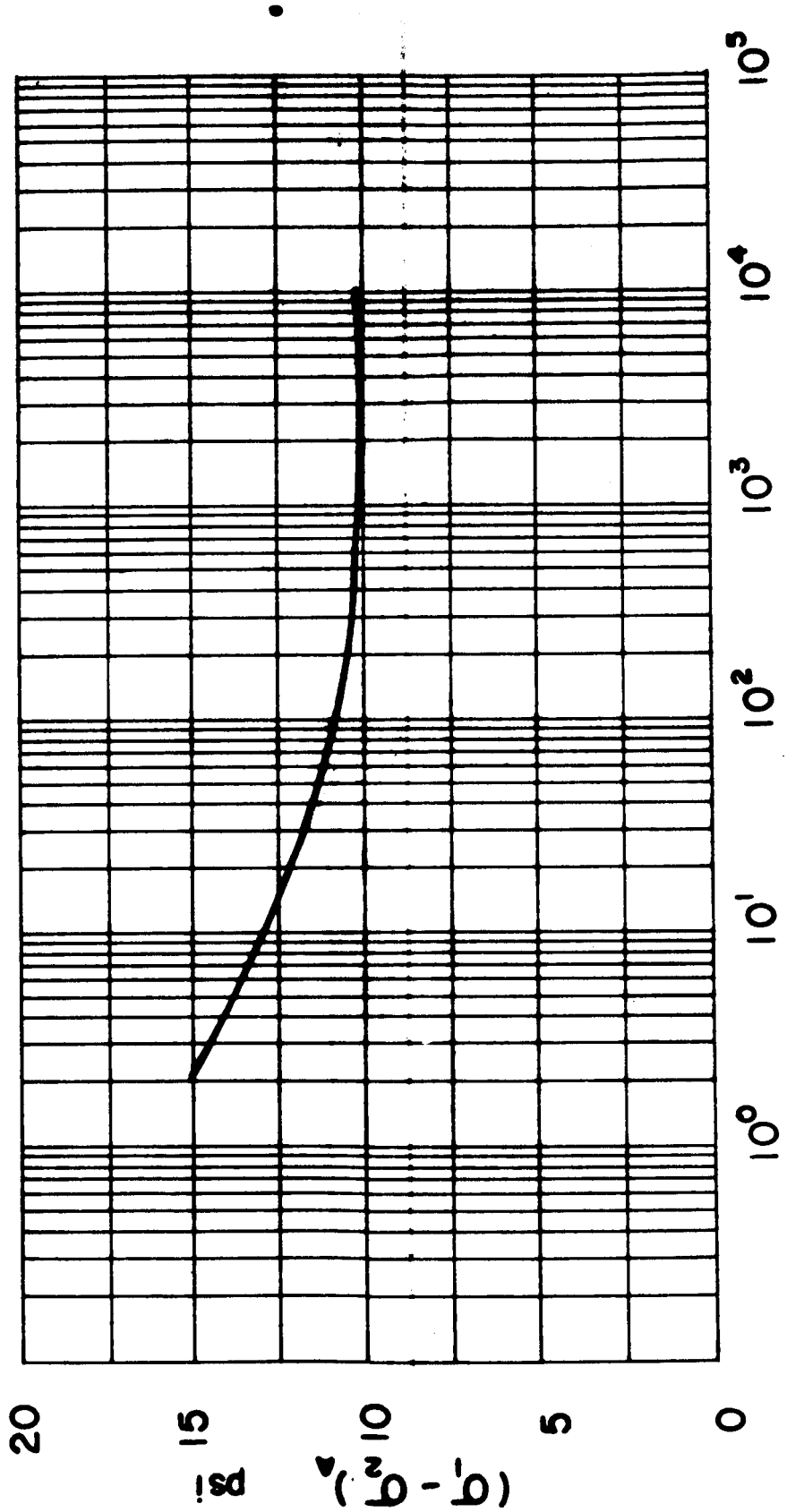
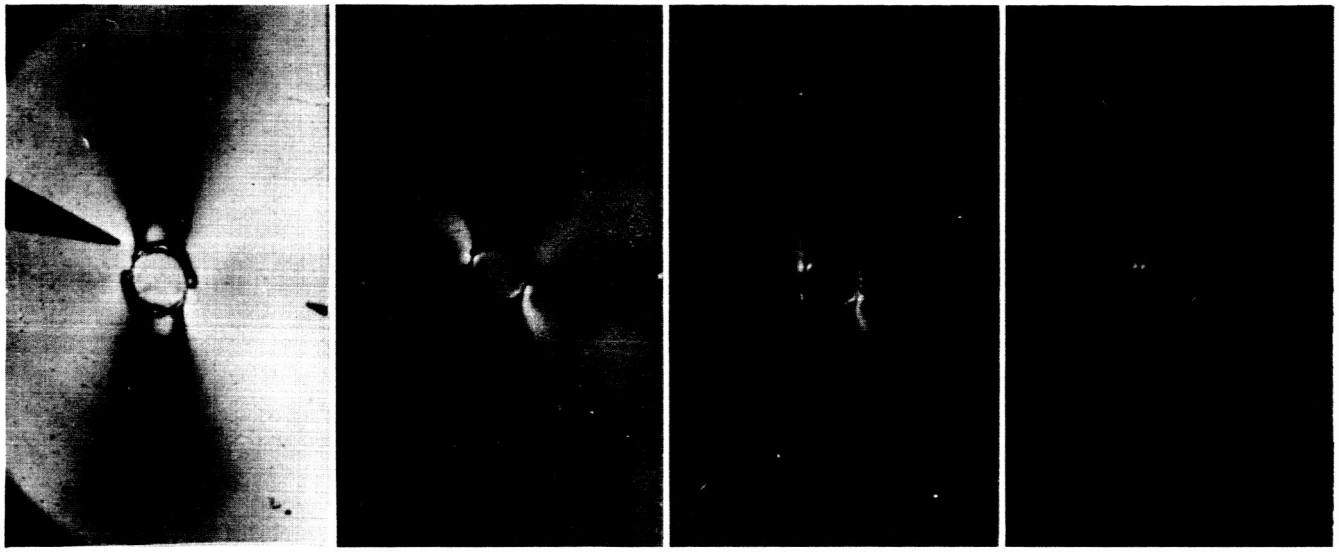
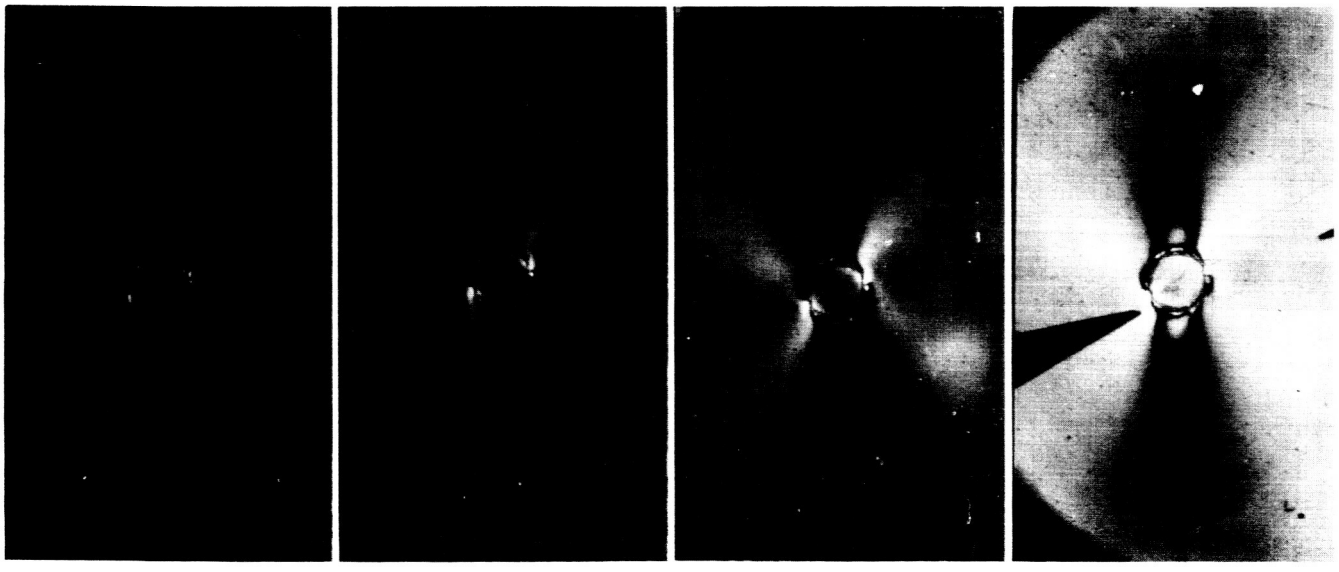


Figure 10. Plate with Bonded Rigid Disc.



$\alpha = -18^\circ$ $\alpha = -8^\circ$ $\alpha = -6^\circ$ $\alpha = -3^\circ$



$\alpha = 0^\circ$ $\alpha = 5^\circ$ $\alpha = 12^\circ$ $\alpha = 22^\circ$

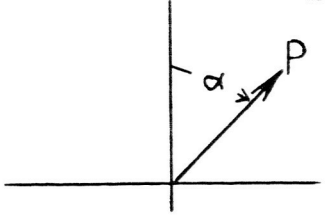
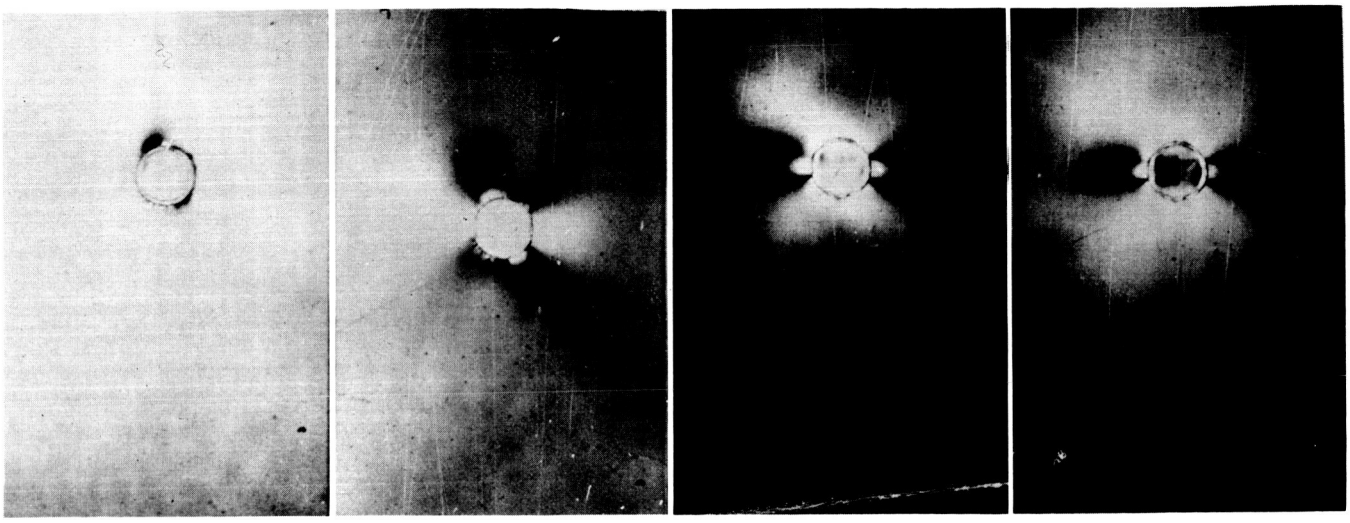
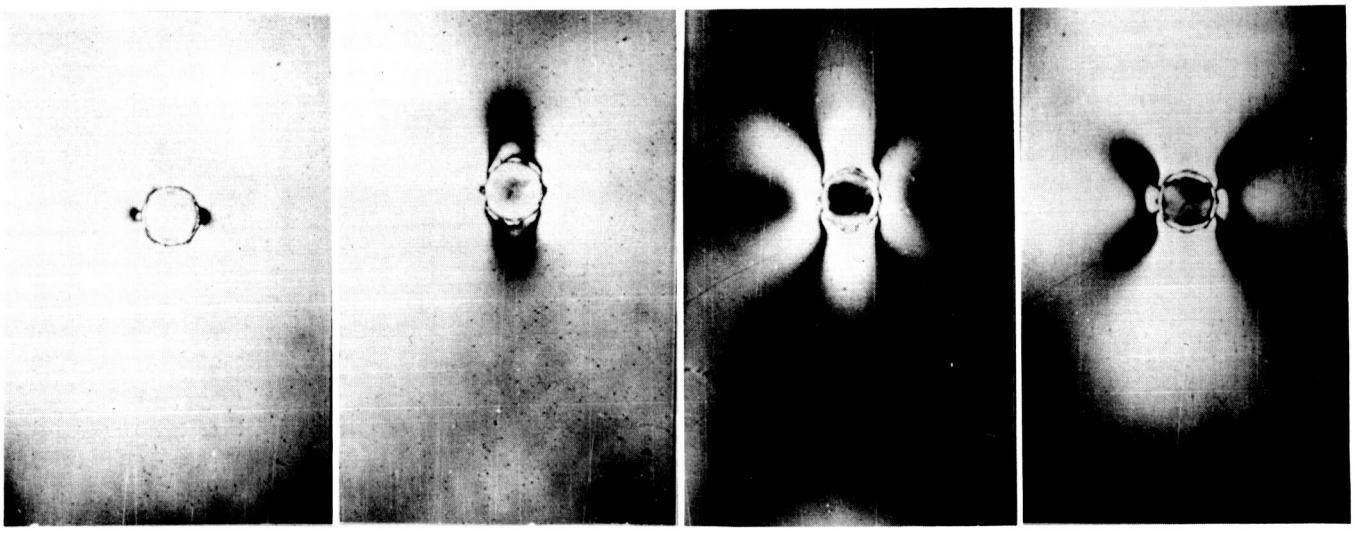


Figure 11. Plate with Bonded Hysol Disc, Isoclinics.



0.2 sec. 25 sec. 90 sec. 114 sec.



350 sec. 1450 sec. 5600 sec. 46,800 sec.

Figure 12. Plate with Bonded Hysol Disc, Isochromatics, Light Field.

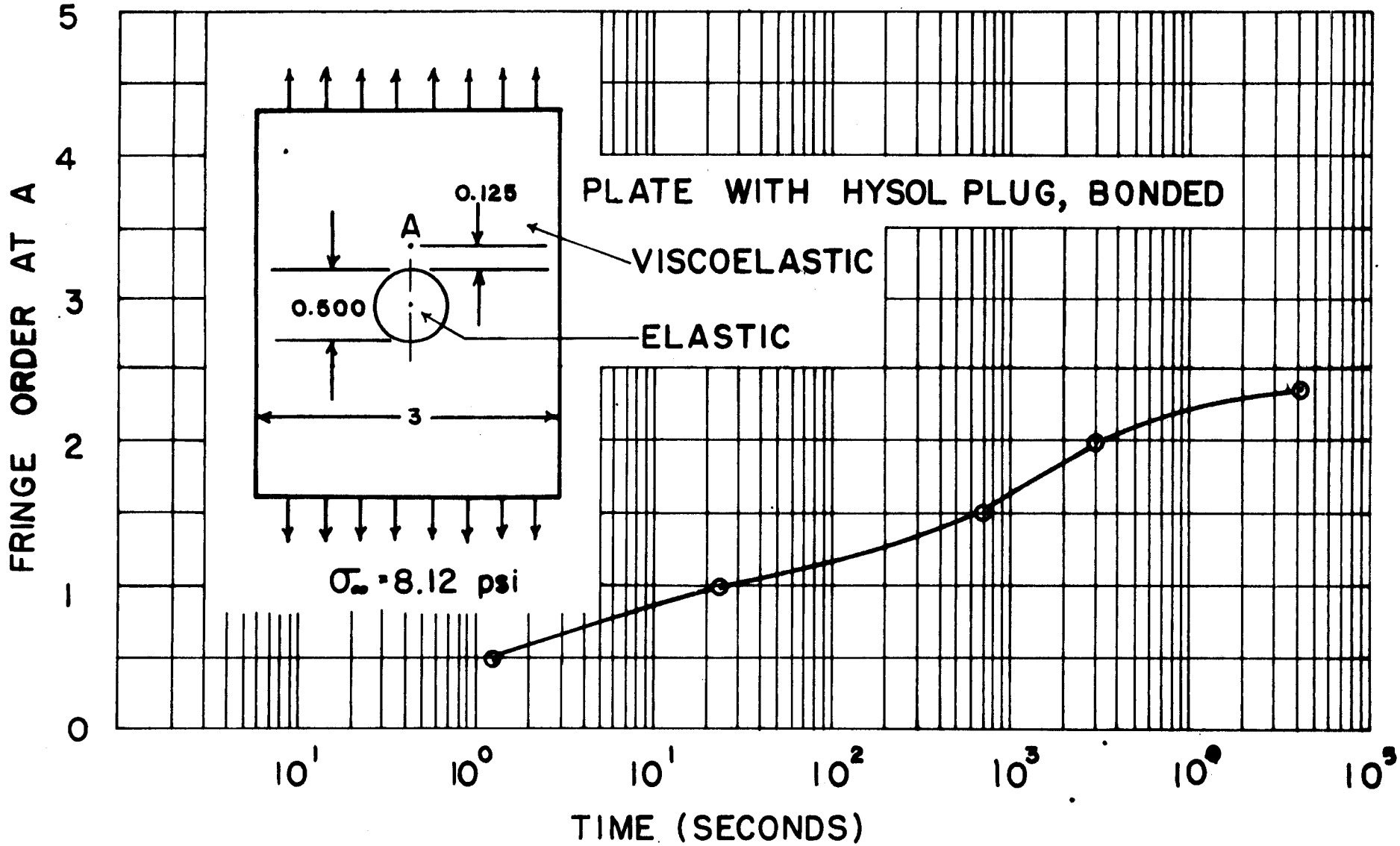


Figure 13. Fringe Order History.

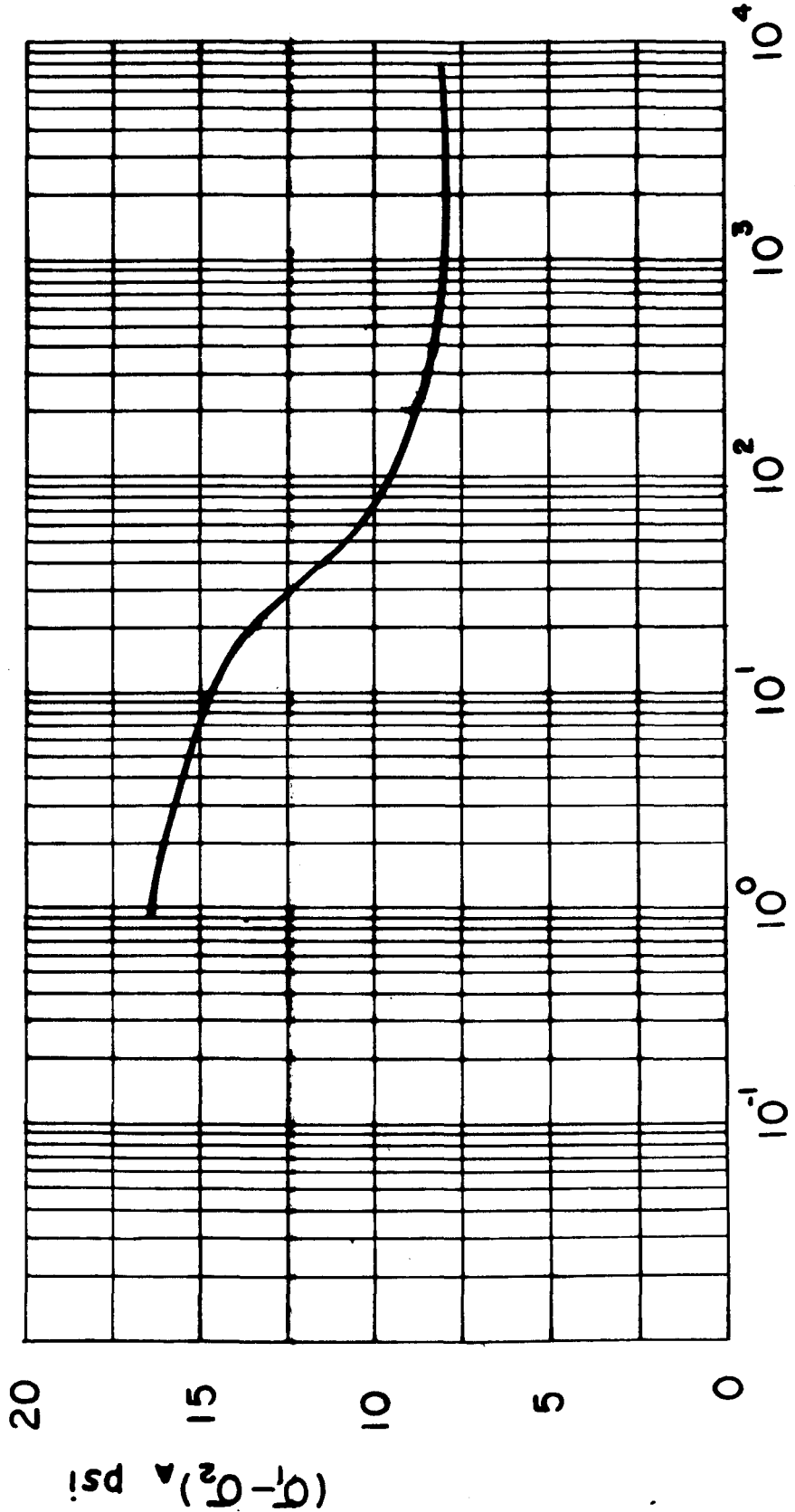
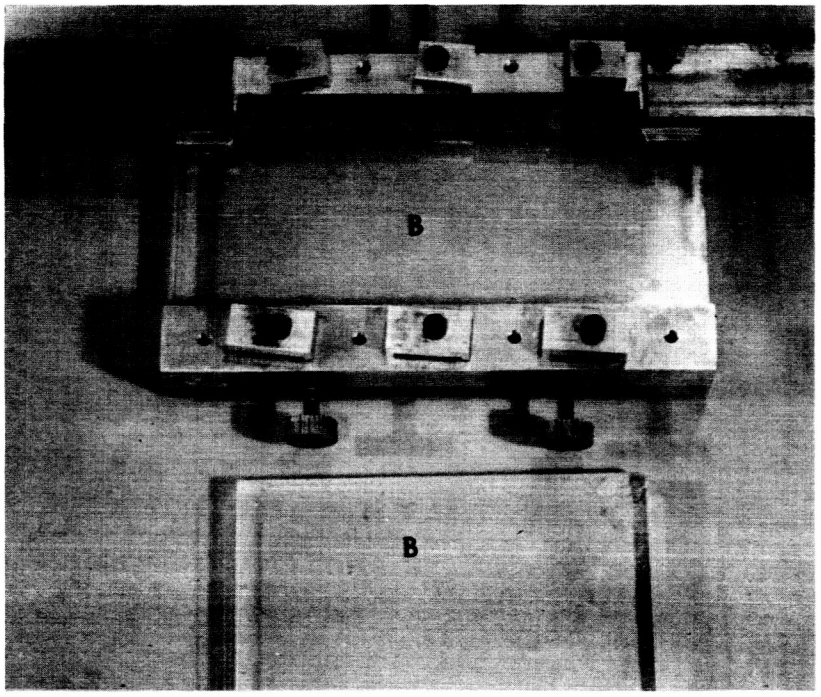


Figure 14. Plate with Bonded Hysol Disc.



A) Latex Diaphragm

B) Plexiglass Plates

Figure 15. Edge Pressure Loading Jig.

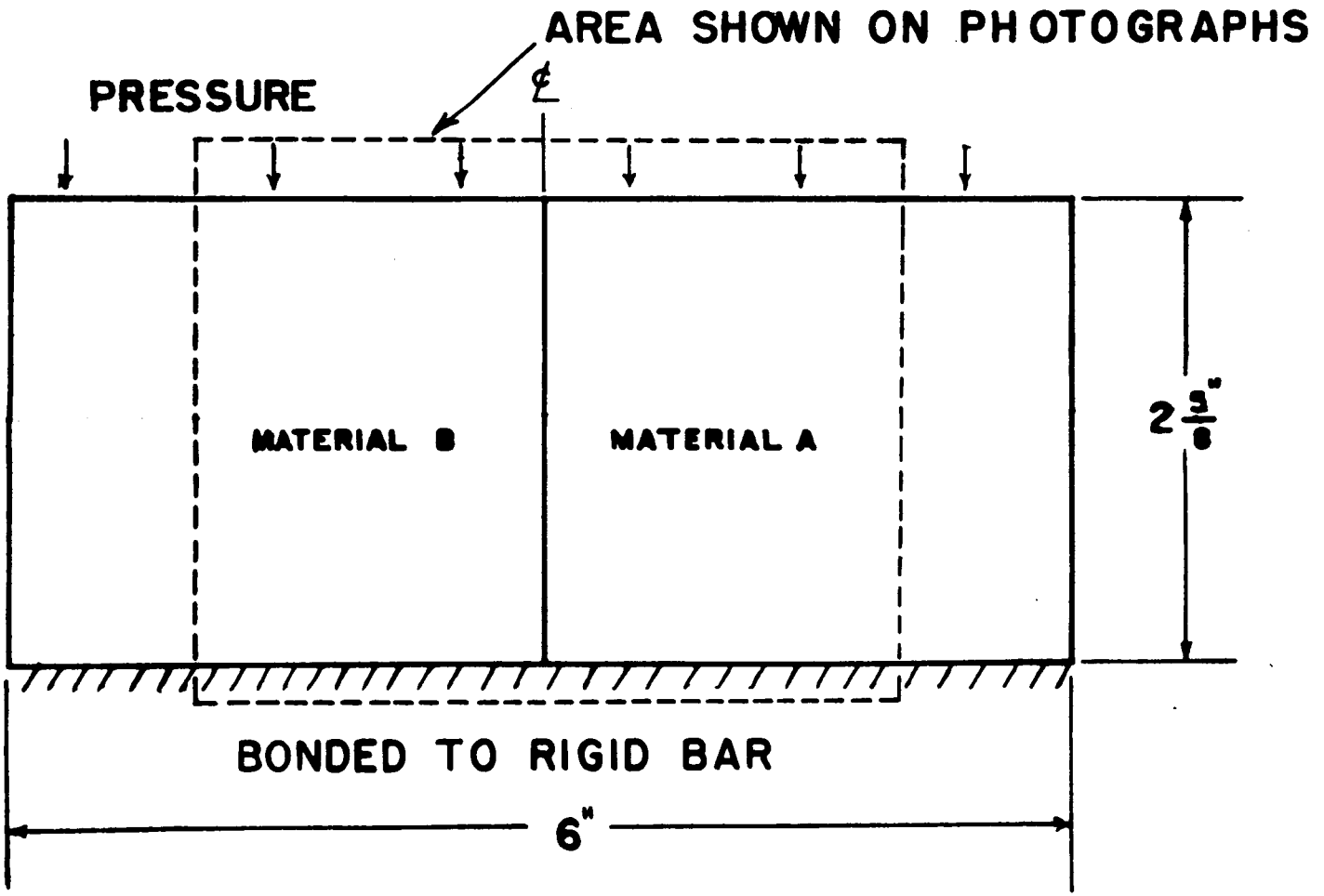
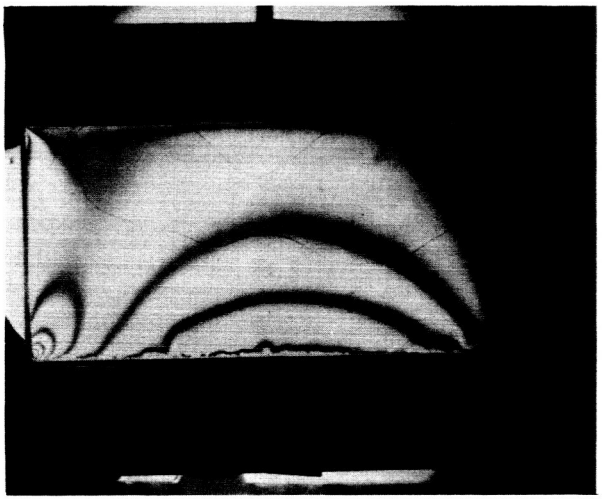
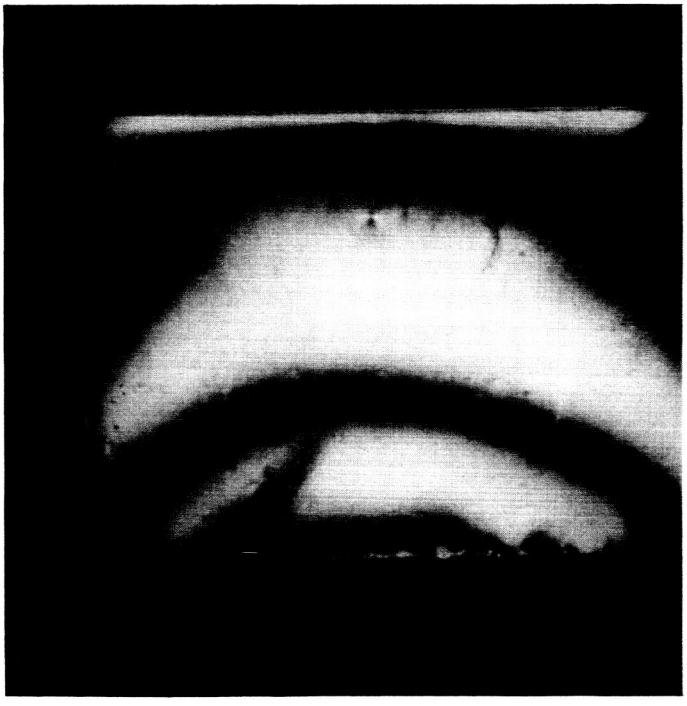


Figure 16. Edge Pressure Loading Jig.



a) Complete Specimen



b) Center Region Enlarged

Figure 17. Edge Loaded Calibration Model.

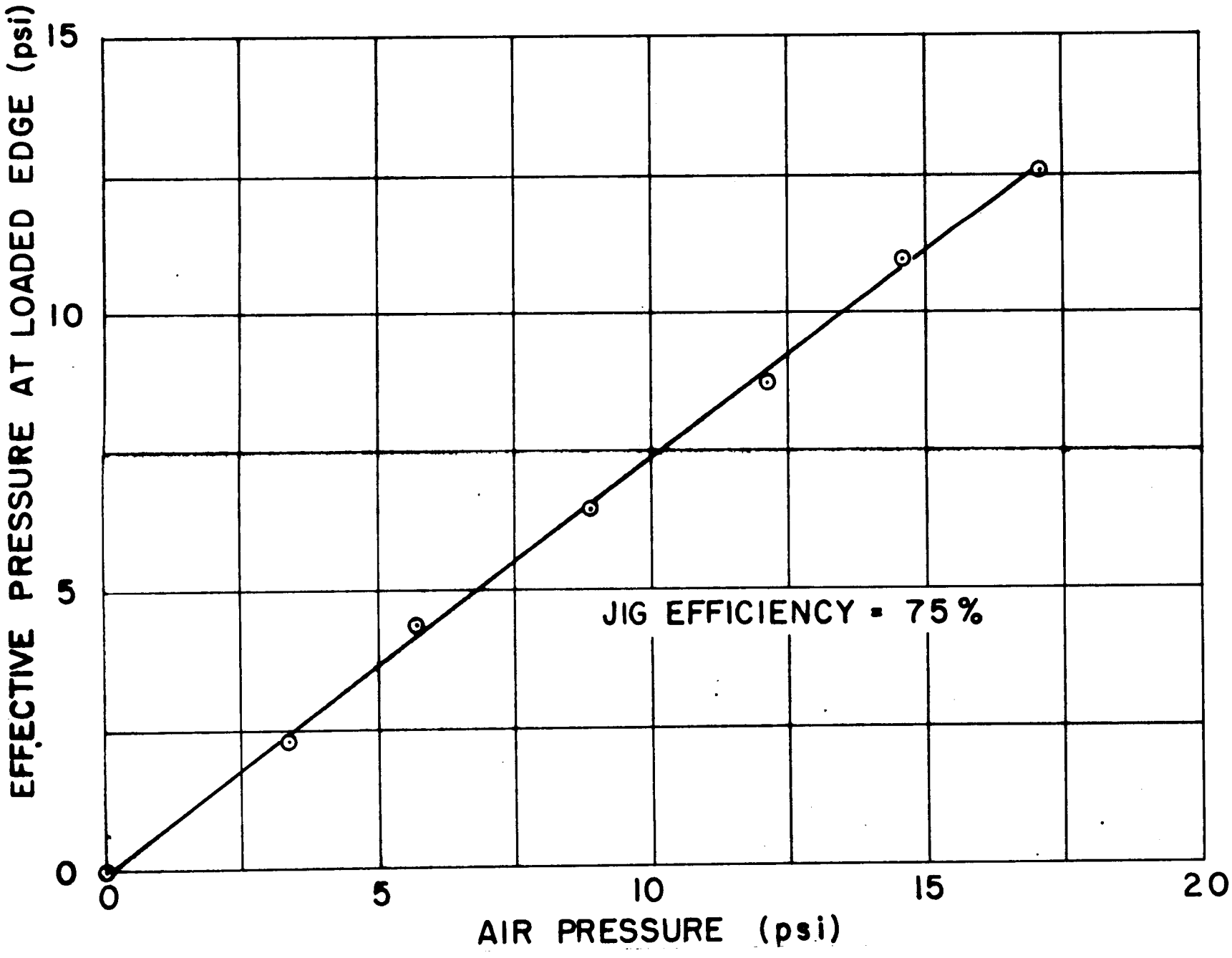


Figure 18. Jig Calibration.

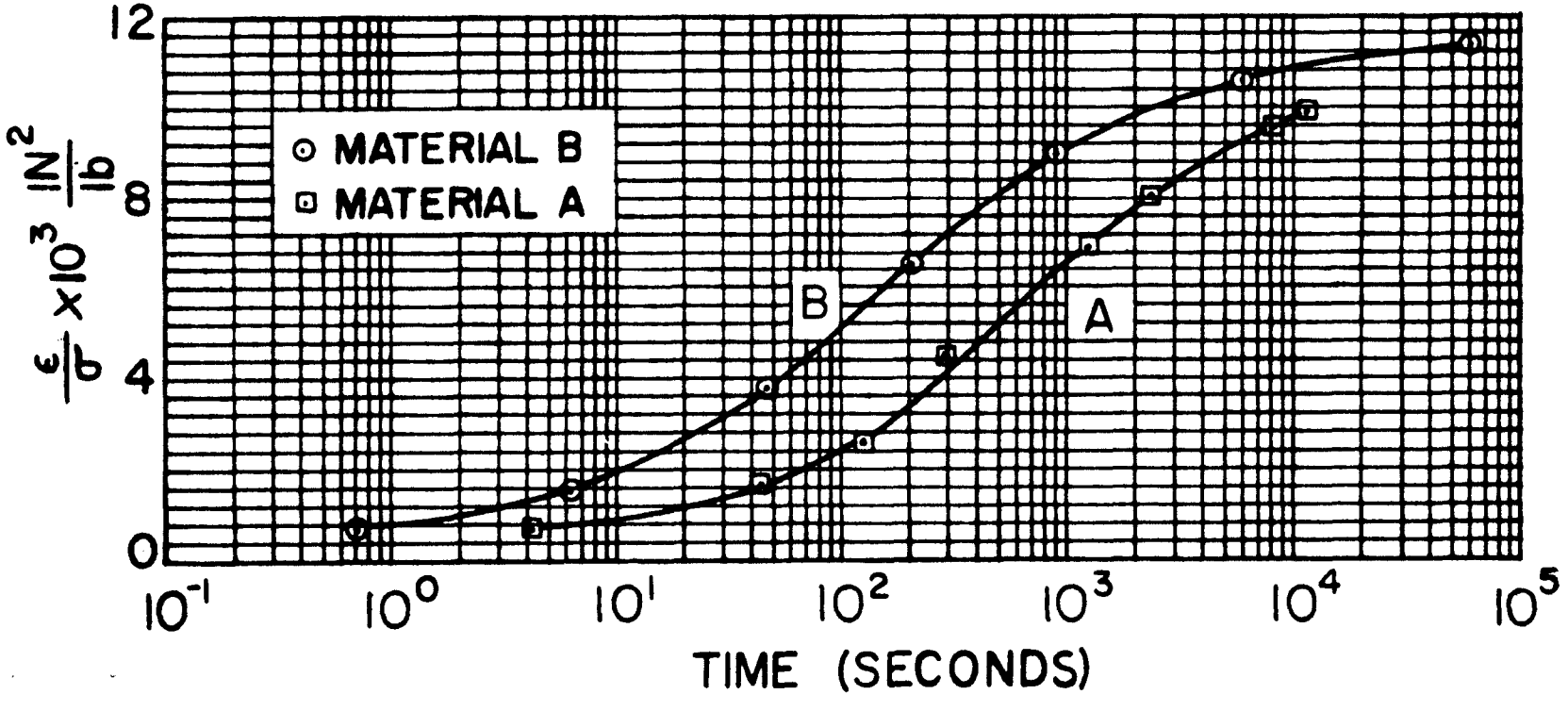


Figure 19. Tensile Creep Compliance.

t = 5 sec. t = 30 sec. t = 10,000 sec. α

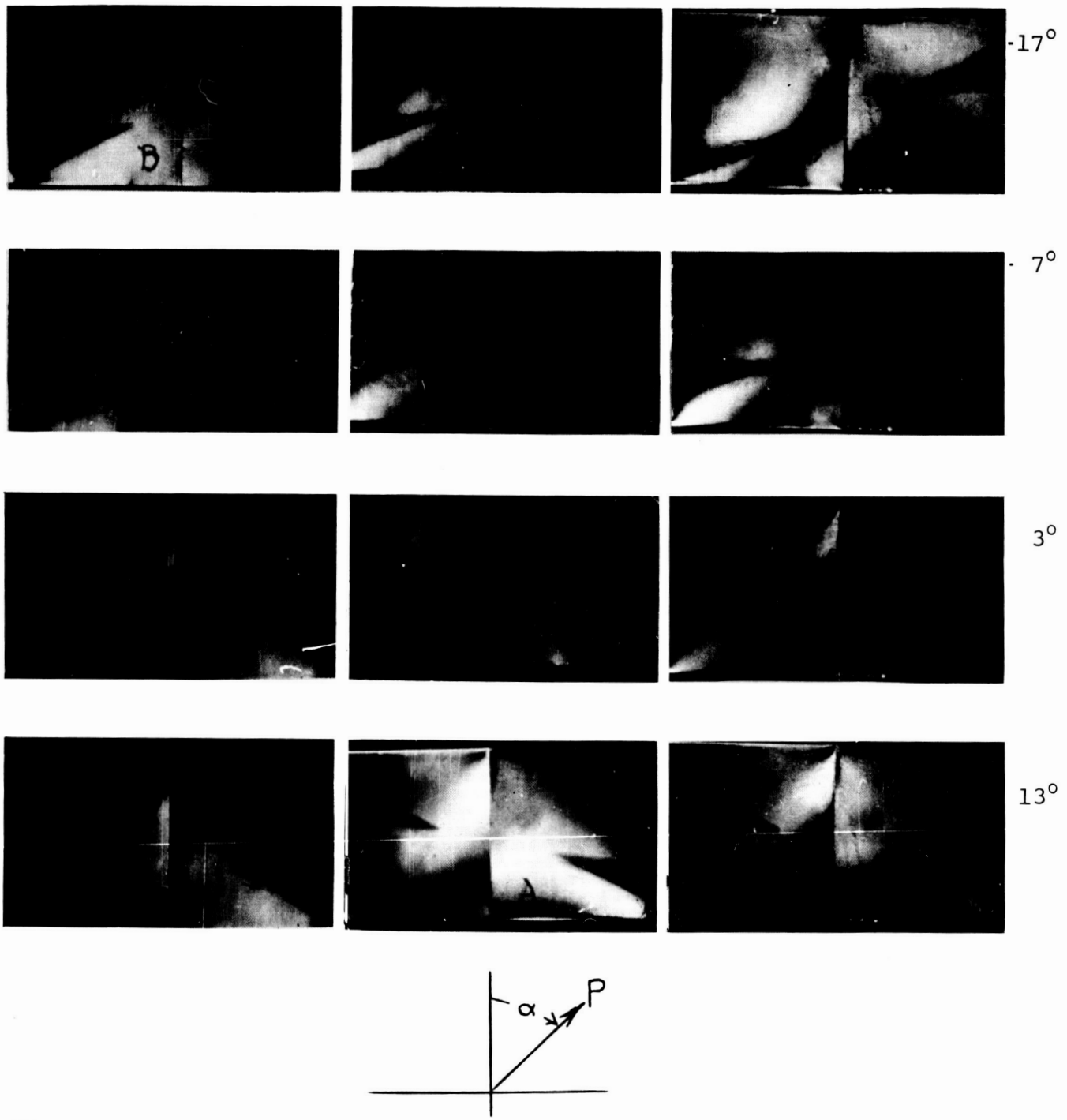


Figure 20. Edge Pressurized, Non-Homogeneous Plate, Isoclinics.

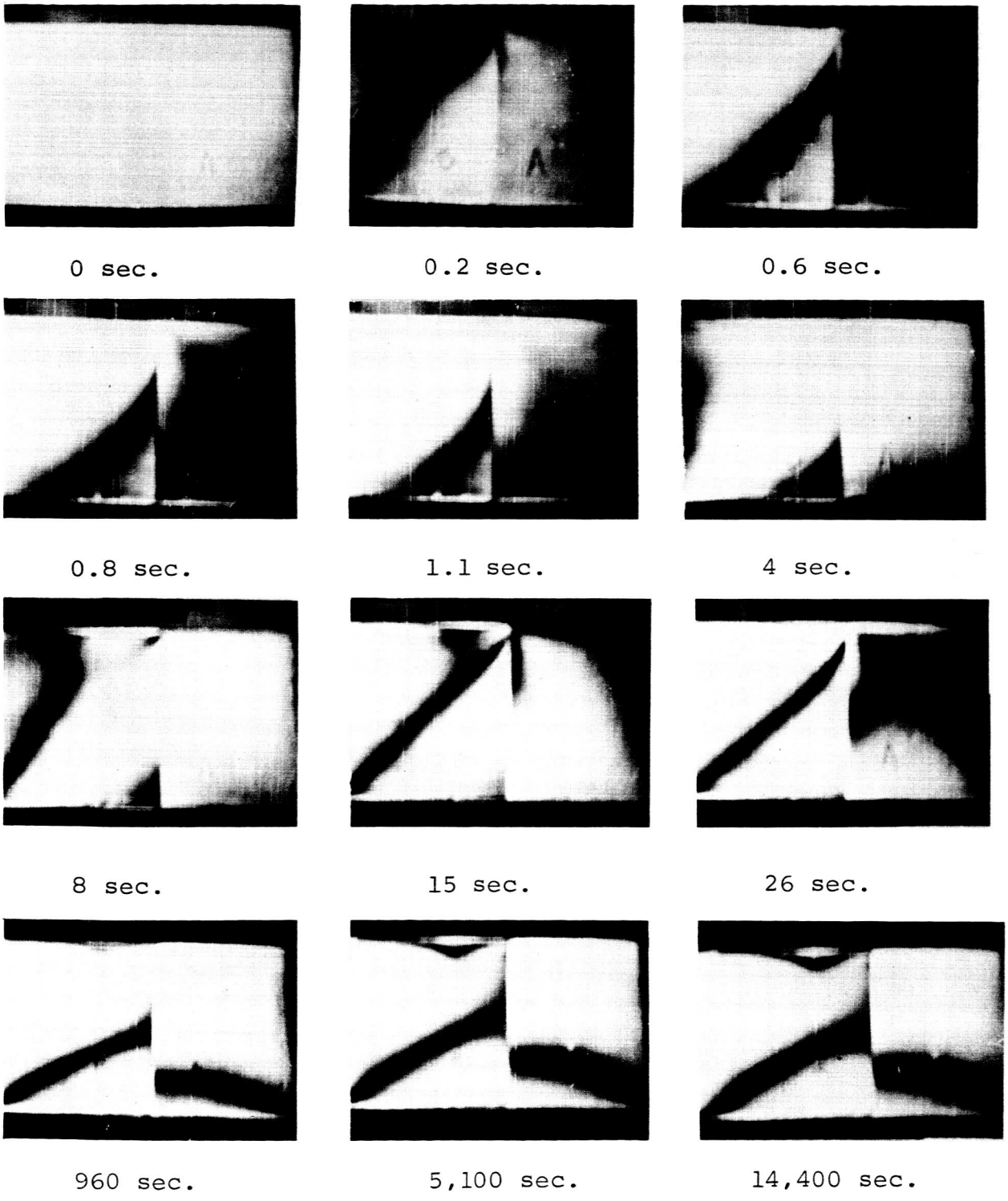


Figure 21. Edge Pressurized, Non-Homogeneous Plate, Isochromatics.

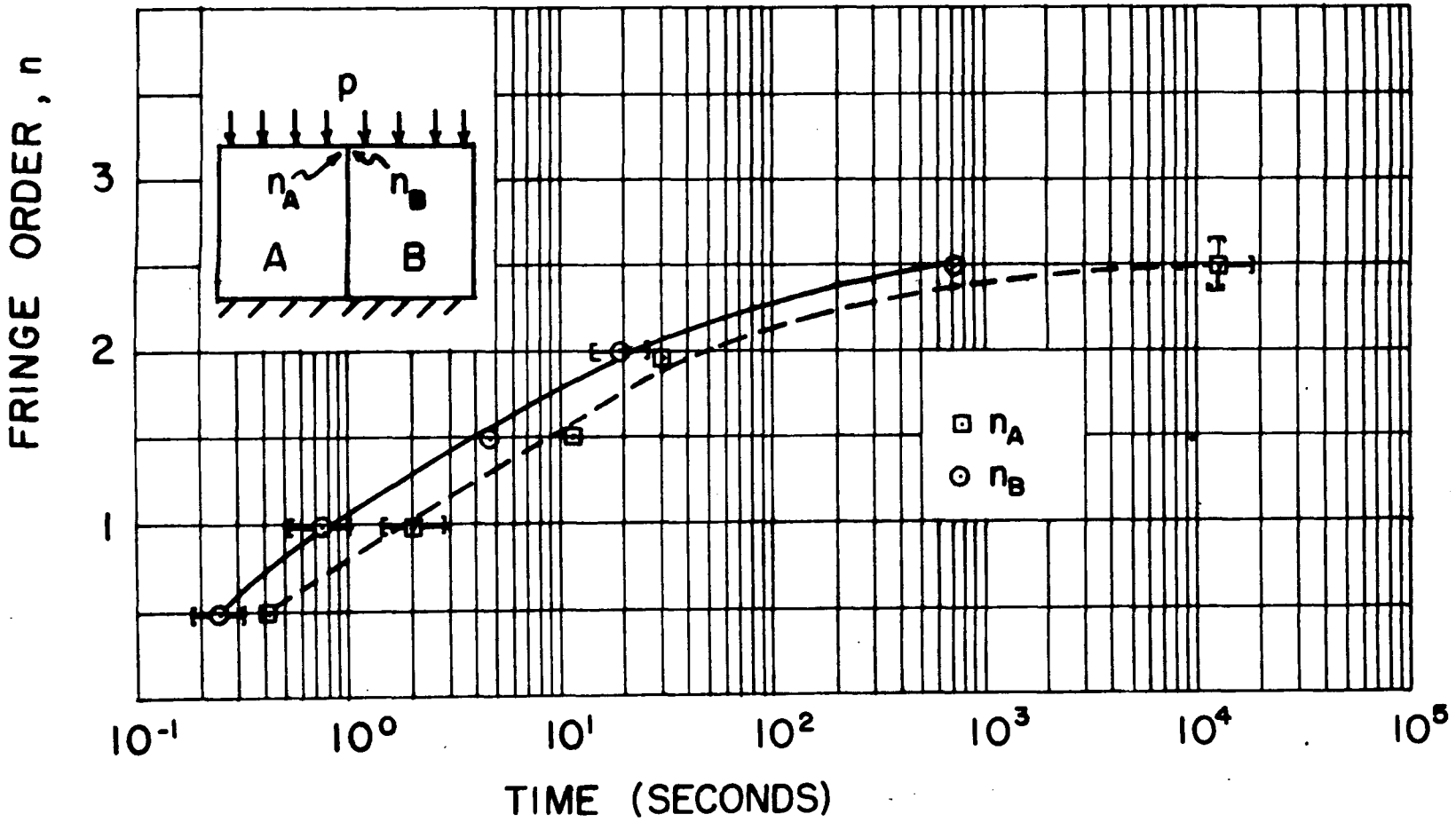


Figure 22. Fringe Order History.

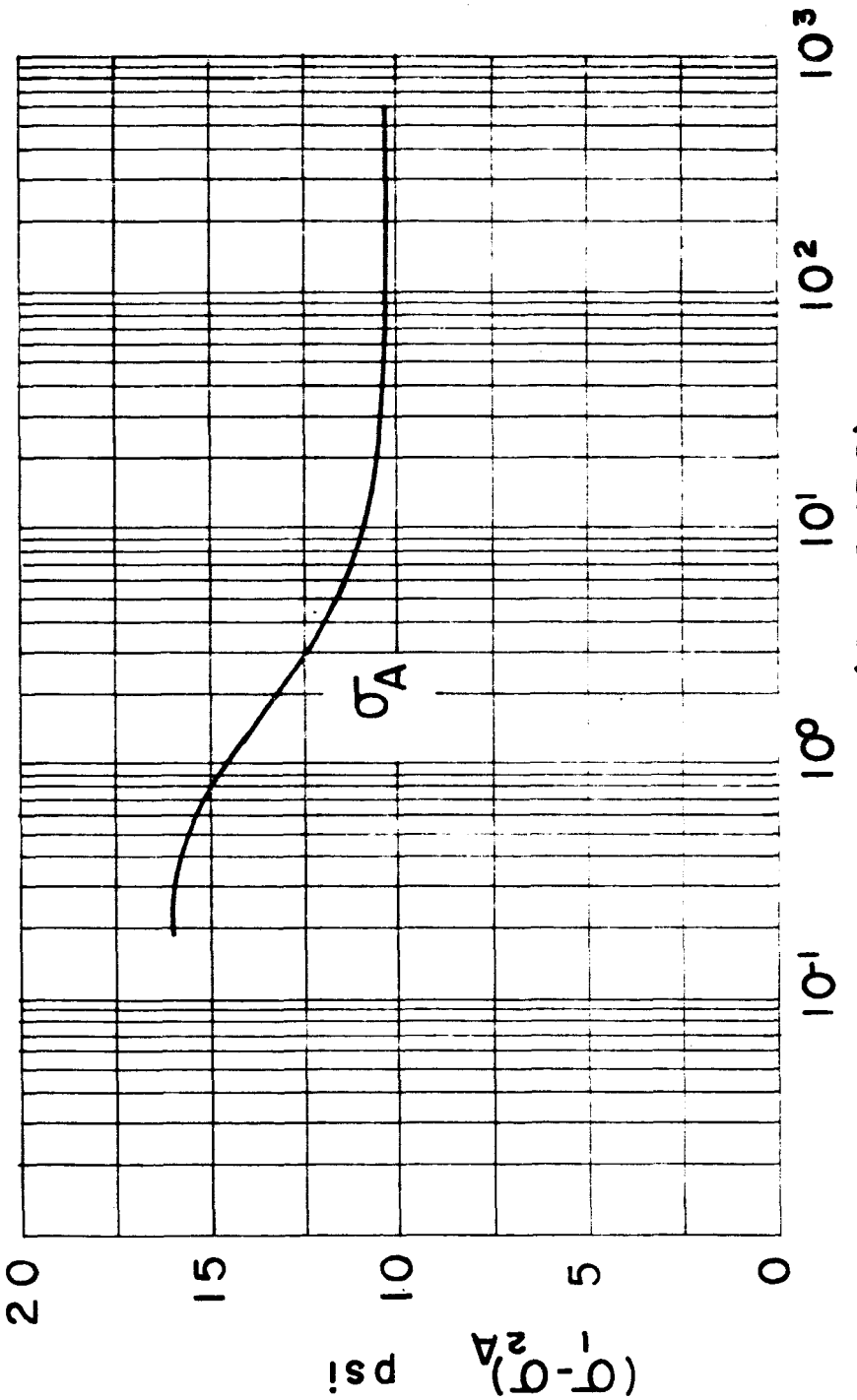


Figure 23A. Stress History, Non-Homogeneous Plate, Material A.

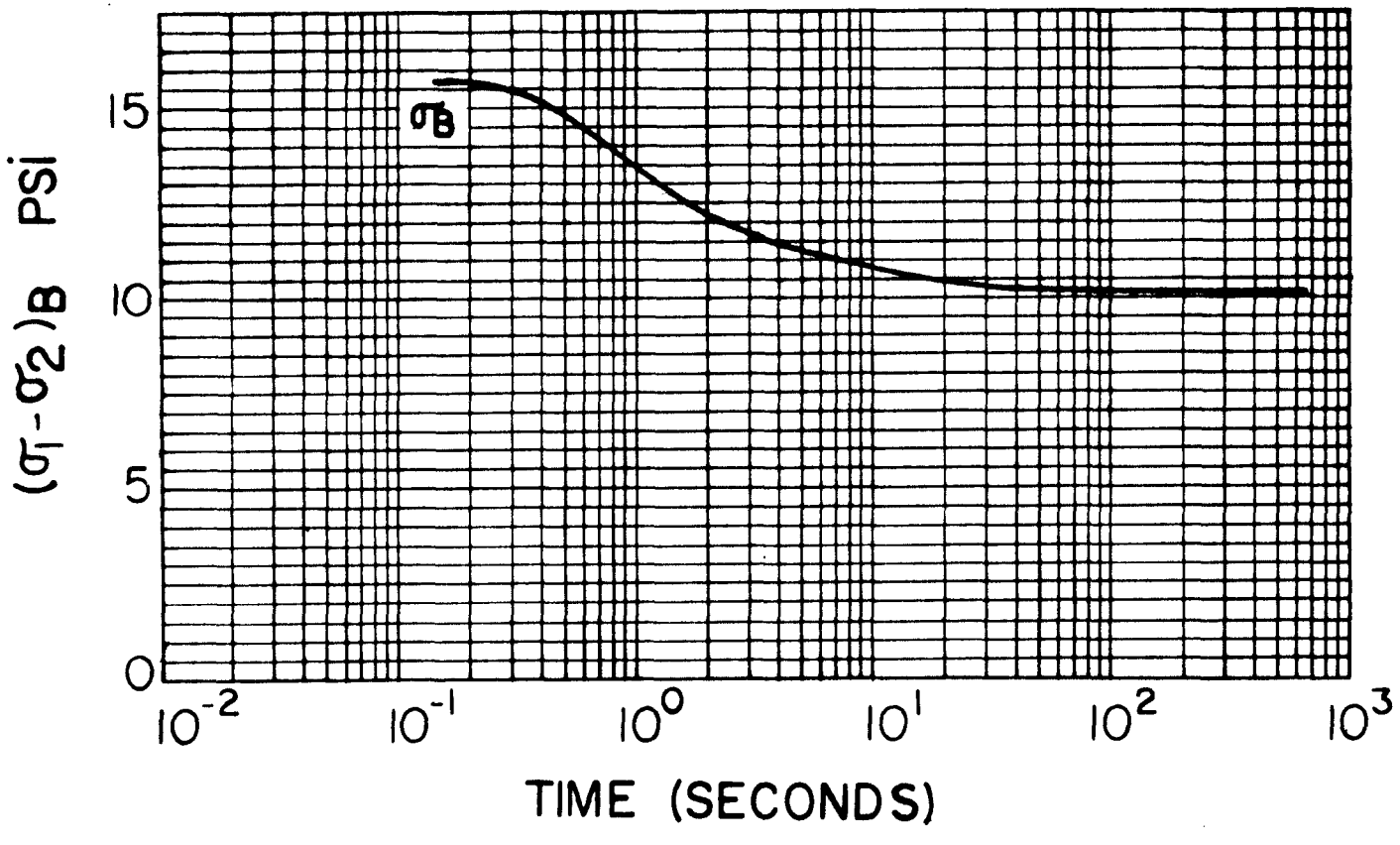
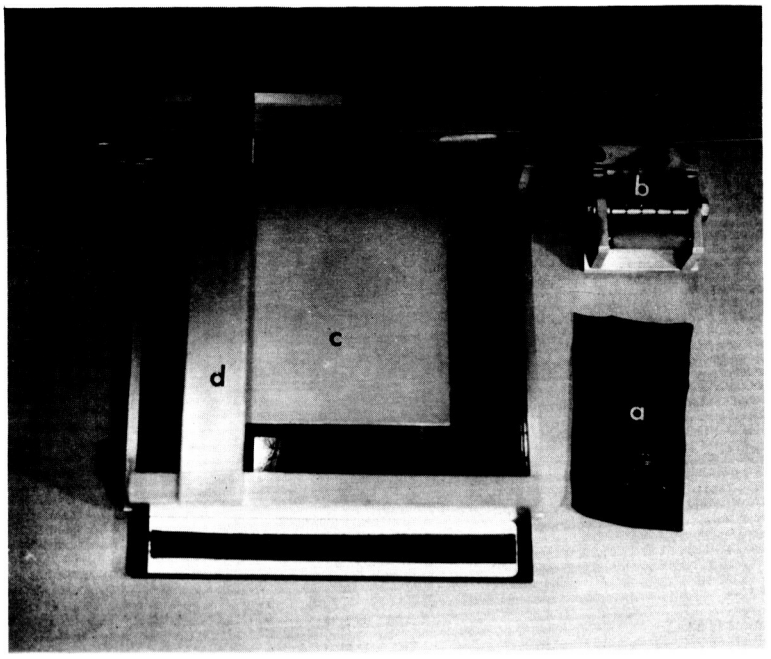


Figure 23B. Stress History, Non-Homogeneous Plate, Material B.



a) Sliced film on rubber sheet.
b) Slicing head.
c) Platen.
d) Indexing T-square.

Figure 24. Polaroid Slicer.

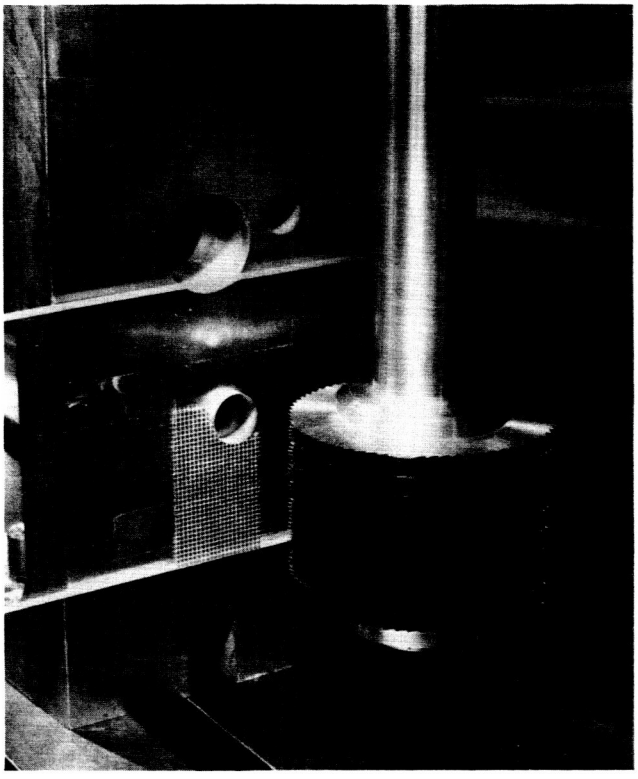


Figure 25. Gang Saw for Slicing Polaroid Film.

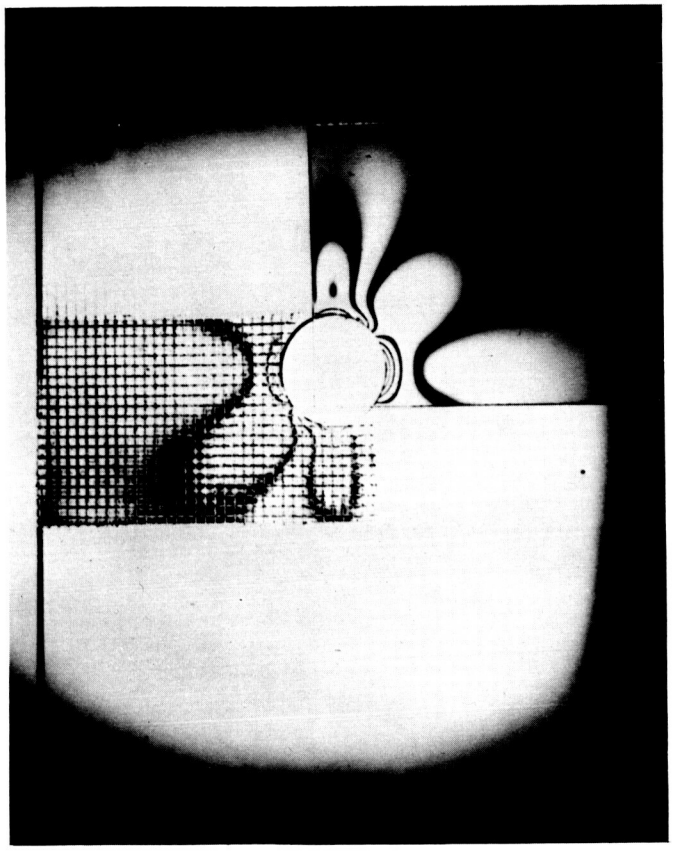
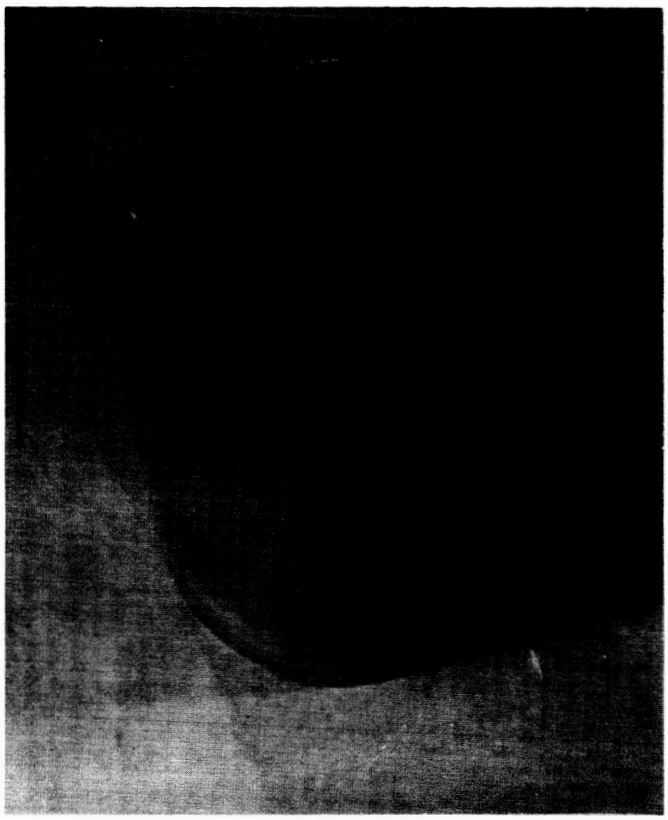
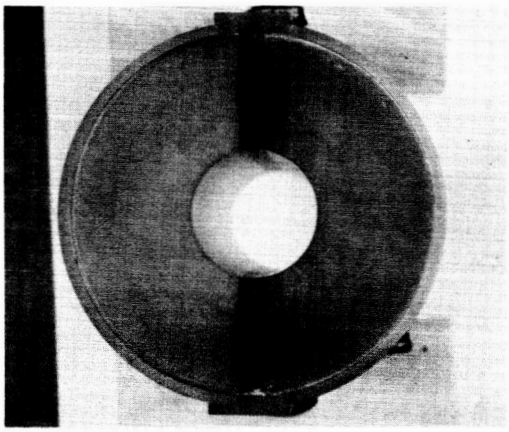


Figure 26. Calibration Model.



a



b

Figure 27. Three Dimensional Model with Imbedded Polariscope.

C. HEATED CIRCULAR DISK

by

T. A. Johnson

The change in stress with time in a circular viscoelastic plate with a circular hole subjected to a steady non-uniform temperature distribution and a uniform pressure distribution on the outer boundary was determined by the analog method.

The loading jig, testing apparatus, and experimental procedure was identical to that described in the last status report[1] and departmental report[2] except as follows:

(1) The resistance wire heater was circular, instead of star-shaped, to conform to the circular port.

(2) The latex diaphragm had no slack, fitting the model like a gasket, barely touching it in the undeformed state. Hence, unlike the previous tests on the star-grain model, the diaphragm had to stretch to follow the deforming material.

(3) An electric time counter was placed over the specimen, allowing the fringe pattern and time to be recorded on the same film. (See Figure 1).

The motivation for this revised design was to facilitate construction of the diaphragm itself and to effect a more uniform

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[1] Dill and Bollard, "Photothermoviscoelasticity Status Report," Department of Aeronautics and Astronautics, University of Washington, July 1965.

[2] Dill and Fowlkes, "Photoviscoelasticity," Report 65-1, Department of Aeronautics and Astronautics, University of Washington, September 1965.

pressure on the outer boundary than had been obtained previously. In comparing with the previous diaphragms, it appeared that the stretching of the latex did not affect the linearity of actual pressure in the model versus gage pressure of the diaphragm. It is also interesting that the pressure efficiency of the jig was 72%, which is in the range of the efficiencies of 68-75% previously obtained.

As Figure 1 shows, the pressure was uniform on the boundary; this is indicated by the concentric character of the fringe pattern.

The model was heated to an axially-symmetric steady-state temperature. The radial variation of temperature is shown in Figure 2. The pressure was suddenly applied to the outer edge. The variation of fringe order at a point on the inner boundary is shown in Figure 3 for two different applied pressures. The stress on the inner boundary was then determined by the analog method (Figure 4).

The stress on the inner boundary was expected to be smaller at short times because of the higher rate of relaxation in the hotter material. The stress should then tend to the elasticity solution. The result has this property.

The purpose of this test was to compare an experimental solution with an analytical solution for some non-trivial thermoviscoelastic problem. The analytical solution is, however, not yet complete.

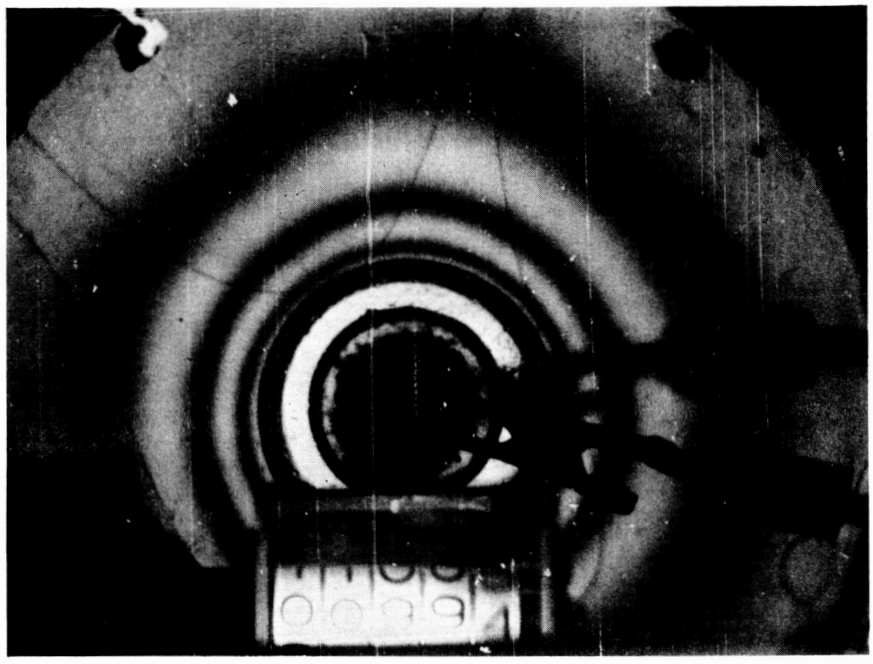

Figure 1. Fringe Pattern for Heated Disk.
(run 2, 99.4 seconds)

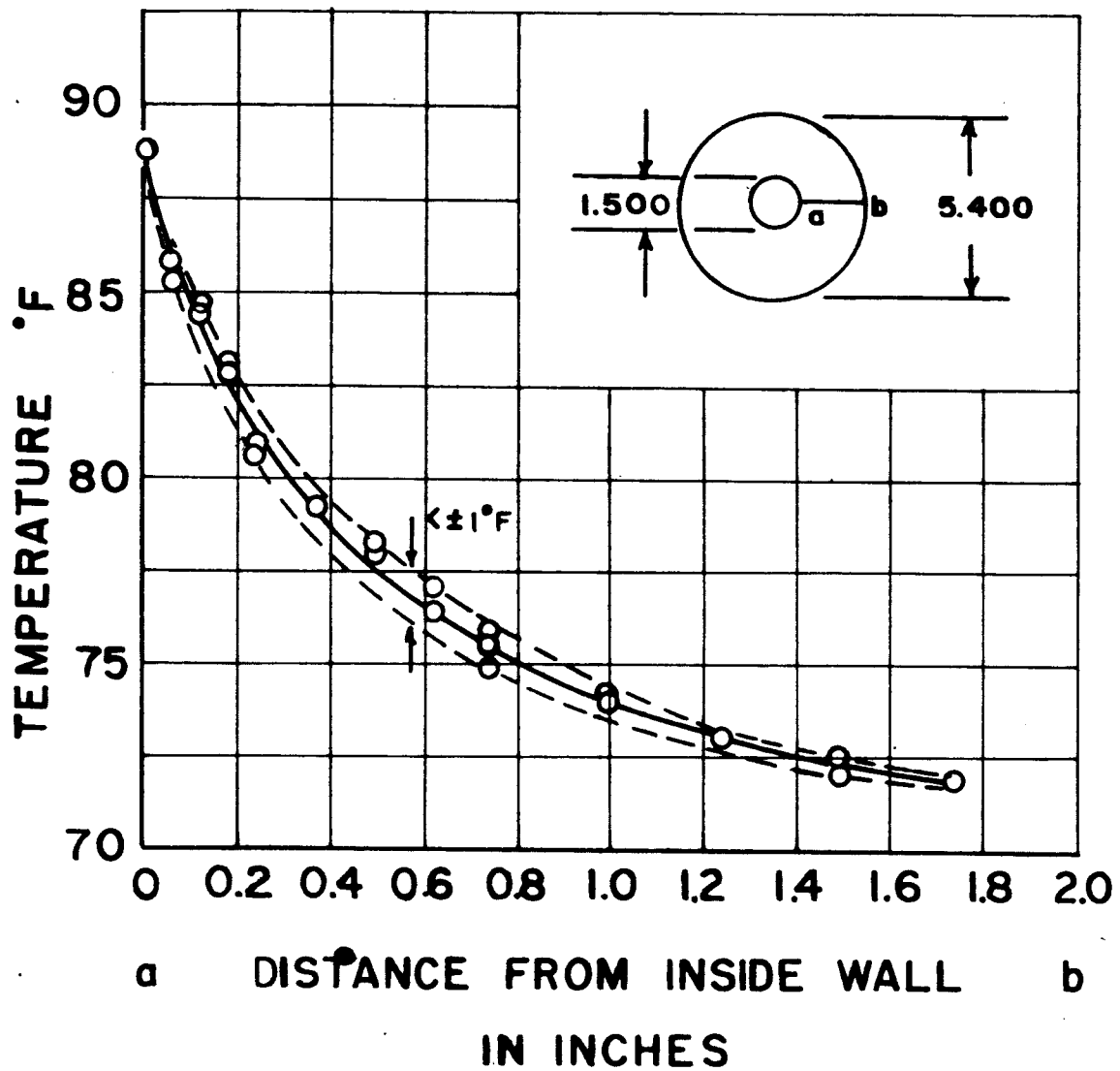

Figure 2. Temperature Distribution.

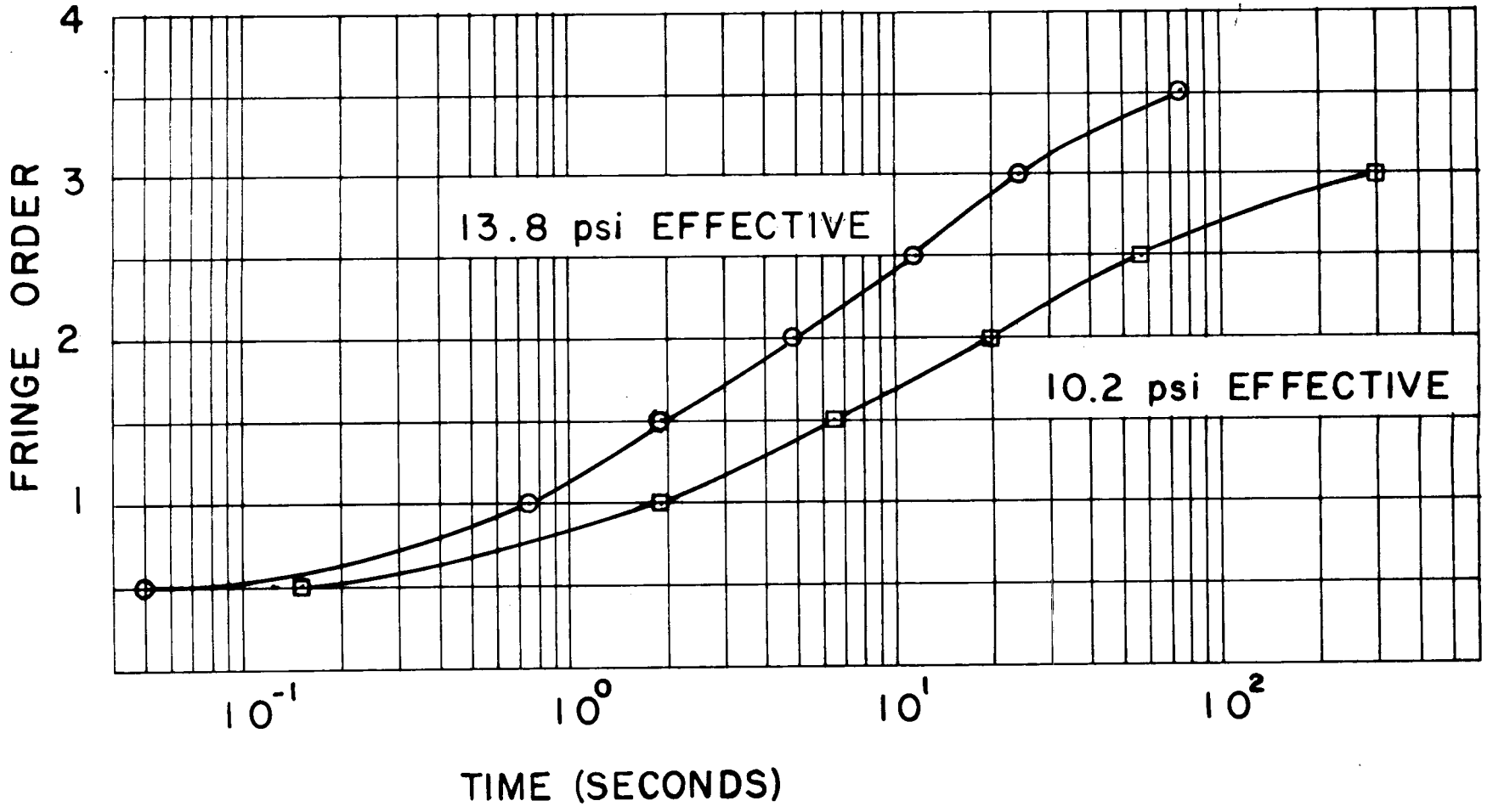

Figure 3. Fringe Order History for Point on Inside of Boundary.

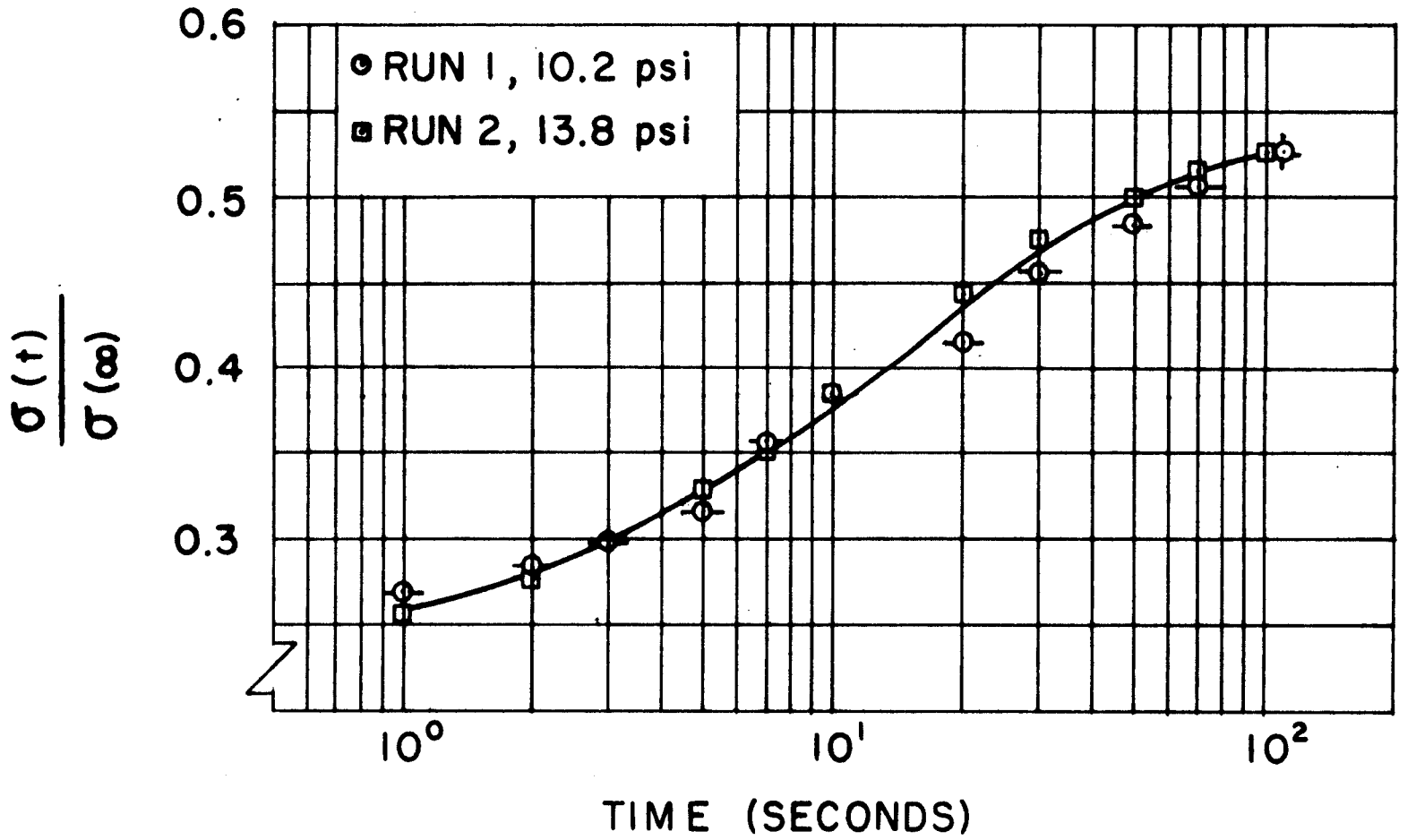

Figure 4. $\frac{\sigma(t)}{\sigma(\infty)}$ History for Point on Inner Boundary at 89°F.

D. FARADAY CELL POLARISCOPE

by

M. E. Fourney

In the general photoelastic or photoviscoelastic problem, the direction of the principal stresses must be determined. This is generally done by observing the isoclinics for various angles while the loading and strain field remain constant. The isoclinic is the locus of points having constant inclination of principal axes of the dielectric tensor. They are obtained by transmitting linear polarized light through the model. By changing the direction of polarization of the light, one can find the direction of the principal axes for all points in the model.

For an elastic material under constant loads, this can easily be achieved by manually changing the plane of polarization of the light. For dynamic loading or viscoelastic stress fields that are slowly varying functions of time, a mechanical rotating element polariscope has been designed and built. This has been described in some detail in the status report of July 1965. For more rapid dynamic loads or for short times, when the stress field for viscoelastic material is a rapidly varying function of time, a Faraday cell polariscope has been designed. The Faraday cell is an electro-optic device; it has the inherent ability to alter the plane of polarization of the observing light at very high speeds.

The Faraday Effect consists of a rotation of the plane of polarization by an applied magnetic field. The rotation is proportional to the path length in the medium and to the component of the

magnetic field in the direction of propagation. The angle of rotation is given by

$$\alpha = VBd$$

where B is the component of the magnetic field in the direction of propagation, d is the path length, and V is the Verdet constant.

Conventionally, positive Verdet constant means that the plane of polarization is rotated, by passage through the medium, in the same sense as the direction of flow of positive electric current flowing in a solenoid which could produce the magnetic field. The rotation is in the same sense for either direction of propagation of the light. Hence, the rotation could be multiplied by reflecting the light several times through the active medium.

When light is transmitted through a medium it is absorbed according to the relation:

$$I(X) = I_o e^{-kx}$$

where k is the absorption coefficient. Both the Verdet constant and the absorption coefficient are functions of the wavelength of light.

A convenient arrangement for a Faraday cell is to place the active material inside a solenoid. The selection of the active material is based on a high value of the Verdet constant but low value of absorption coefficient. The material that has been selected is a high density lead glass. The constants of this glass are given by the supplier as follows:

$$V = 0.109 \text{ min/cm - gauss},$$

$$I/I_0 = 97.5\% \text{ /in,}$$

$$\text{For } \lambda = 7000A^{\circ}.$$

Two glass rods 10 inches long with a square cross-section one inch on a side were obtained and the ends polished. A solenoid was wound to supply the required magnetic field. Figure 1 shows both this solenoid and the glass rod. The cell was found to give 2.2° of rotation per ampere for the mercury green line.

A small amount of residual strain was observed in the rod when placed in a polariscope; however, it is not large enough to cause serious distortion of an image that is transmitted through the rod.

The solenoid has been constructed in a manner to keep the inductance low so that the switching of the cell could be rapid. The inductance of the solenoid is 14.7 millihenries.

A polariscope is under construction that will utilize the Faraday cell as an element to rotate the plane of polarization of the light. The purpose and general construction features will be similar to the rotating element polariscope previously discussed. The major difference will be the speed of rotation. In the Faraday cell polariscope, the only limitation is the speed with which the magnetic field of the solenoid can be altered.

A general schematic diagram of the Faraday cell polariscope is shown in Figure 2.

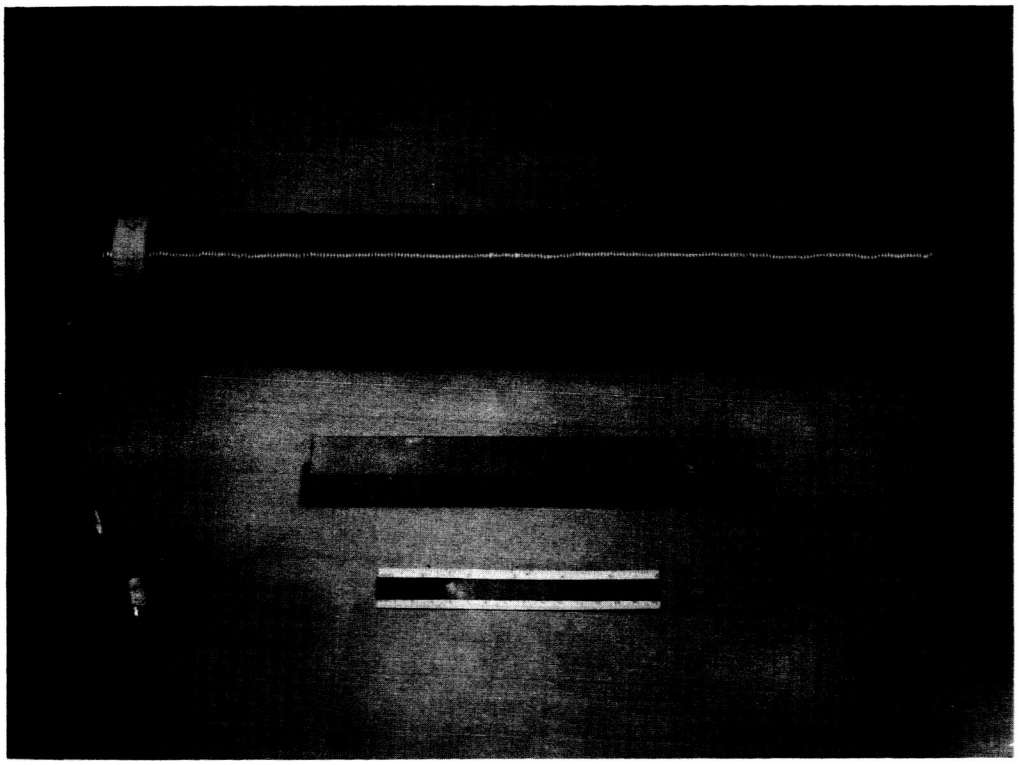

Figure 1. Faraday Cell Glass Rod and Solenoid.

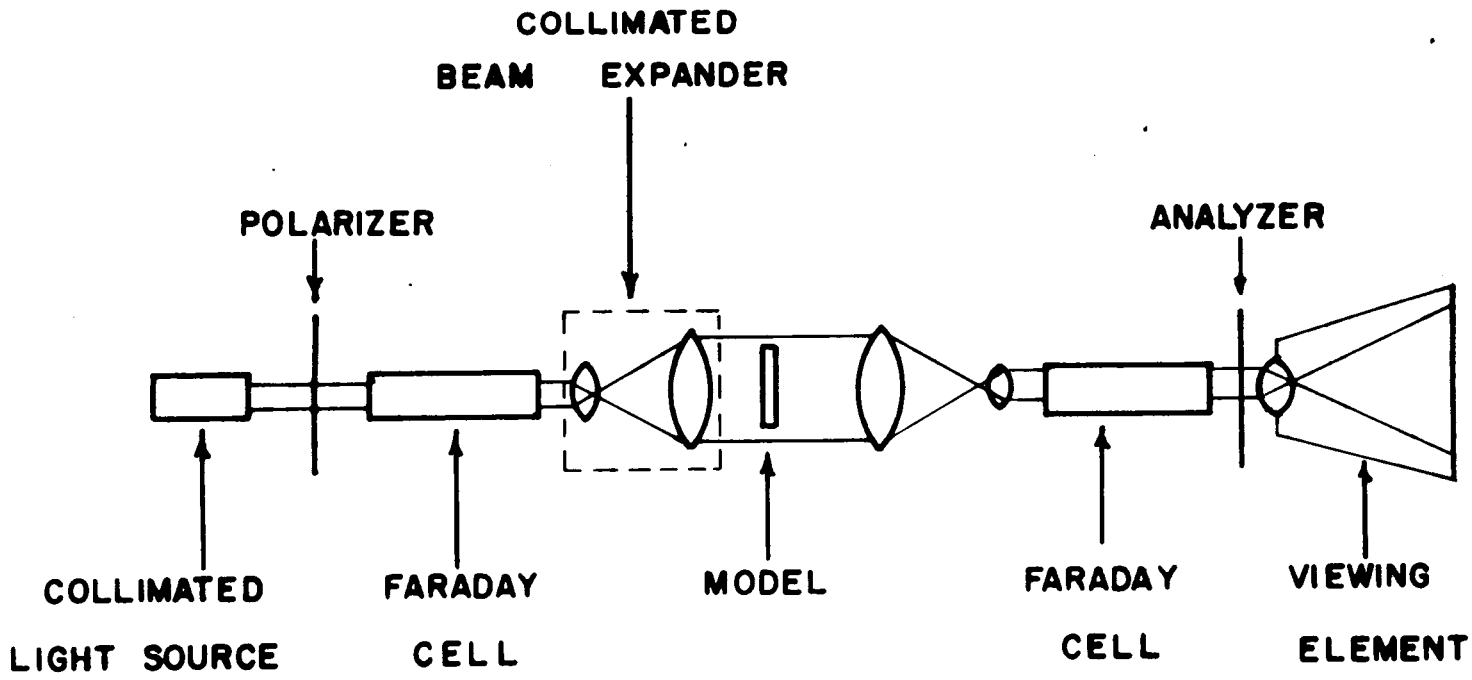

Figure 2. Schematic of Faraday Cell Polariscope.

E. NON-LINEAR PHOTOMECHANICS

by

P. Ramanaiah

1. INTRODUCTION

One of the main problems of a structural engineer is the prediction of the strength and deformation characteristics of various structures made of different types of materials. With the tremendous advancement of industry, more and more materials of widely differing characteristics are being used as structural materials. These materials cannot be adequately treated by the usual linear elasticity theory. For example, rubbers used in industry behave non-linearly in their stress-deformation characteristics. High polymers, used extensively as part of solid propellants for rockets, have time dependent mechanical behavior. In order to keep pace with the industry, the modern engineer is forced to deal with such materials.

The stress analysis of a structure, even with simple constitutive equations such as classical linear elasticity, becomes very complicated if the shape of the structure is irregular. If the constitutive equation is non-linear, the problems of stress analysis almost defy any analytical solution except for some very simple body shapes and types of loading. In such cases, it is desirable to have an experimental procedure to find the stresses in the body. There are several such methods, but we are mainly concerned with the generalization of the method of experimental stress analysis known as photoelasticity.

When certain transparent materials are strained, they become doubly refractive and therefore exhibit fringe patterns when placed between polaroids. Such materials are called birefringent. Photomechanics is based on the assumption that the birefringence exhibited by a material depends upon the strain it has undergone. The solution of a stress analysis problem can be determined experimentally in the following way. A model is constructed from the birefringent material and loaded. The birefringence is recorded. The relation between birefringence and strain or stress then gives information about the solution to the problem.

This procedure has been used for some time for the solution of linear elasticity problems [1].[1] More recently the solution of linear viscoelasticity problems has been obtained in this way [2]-[9]. There were also attempts to extend the method to the case of materials in an elastic-plastic state. See [10]-[12].

The theory presented by Mindlin [2], Read [3], and Dill [4] presumes the mechanical-optic relation to be similar to the stress-strain relation for viscoelastic materials. Their work is limited to small displacement gradients. The investigations made by Frocht and Thomson [10] and Monch and Loreck [11] are mainly experimental and they are marked by the absence of any theoretical investigation of the rheo-optic relation[2] used. It is not at all obvious how the

[1]Numbers in square brackets indicate the references listed at the end of the report.

[2]The terms rheo-optic relation, photo-mechanical relation, and mechanical-optic relation are used synonymously. By this we mean a constitutive equation relating stress or deformation tensors to the refraction or related tensors.

rheo-optic constitutive equations obtained by them from simple experiments can be generalized to more complicated situations, especially when the equations are non-linear. The work of Tokouka [12] is theoretical in nature and is based on the proposition that the dielectric constants of a solid depend upon elastic and plastic strains of the material. We believe that this proposition is very restrictive and the resulting theory has limited, if any, application.

Any rational extension of the existing methods of stress analysis by photo-methods to a more wider class of problems can be achieved only by establishing a rigorous general theory of photo-mechanical constitution equations. The aim of the present article is to establish such a theory. We shall make use of the methods recently used successfully in the theory of non-linear mechanical constitutive equations, especially the concept of fading memory first proposed by Coleman and Noll [13]. We derive, in Chapter 5, a very general photo-mechanical constitutive equation satisfying the proper invariance requirements. The existing linear theories of photo-mechanics (photoelasticity and photoviscoelasticity) are shown to be special cases of the general theory as presented in this article. A theoretical basis for non-linear photoviscoelasticity and non-linear photoelasticity is therefore established. Finally, we show that the theory of photoplasticity as was presented in References [10] and [11] is no more than non-linear photoelasticity.

2. GENERAL PRINCIPLES

2.1 KINEMATICS [15]

The motion of a material continuum can be described by specifying the motion of each of its particles. Consider a material particle X occupying position $\underset{\sim}{X}$ in the reference configuration.[3] As time proceeds, this material particle occupies different positions in space. Suppose it occupies positions $\underset{\sim}{X}$ and $\underset{\sim}{\xi}$ at times t and τ ($\tau<t$) respectively, where t is the present time. The coordinate systems used to describe the configurations at different times may be different and curvilinear in general. Then the motion of this generic particle X can be described by

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{X},\tau). \qquad (2.1:1)$$

The gradient of $\underset{\sim}{\xi}$ with respect to $\underset{\sim}{X}$ is called the deformation gradient at the material point X and time τ

$$\underset{\sim}{F}(\underset{\sim}{X},\tau) = \nabla_{\underset{\sim}{X}}\, \underset{\sim}{\xi}(\underset{\sim}{X},\tau), \qquad (2.1:2)$$

where $\nabla_{\underset{\sim}{X}}$ indicates gradient with respect to $\underset{\sim}{X}$. It is important to note here that the deformation gradient not only depends on the configuration at time τ, but also is a function of the reference configuration.

In the mechanics of continuous media, one of the fundamental principles is the permanence of matter. According to this principle, no region of finite positive volume is deformed into one of

[3] We will use the symbol "~" under letters to denote vectors and tensors. The words tensor and "linear transformation from a three-dimensional Euclidean vector space into itself" are used synonymously throughout this paper.

zero, infinite or negative volume. For this condition to be true, it is necessary and sufficient that

$$|\det \underset{\sim}{F}| > 0 \ . \qquad (2.1:3)$$

In other words, for any motion that does not violate the principle of the permanence of matter, the matrix of the deformation gradient is nonsingular at all times. This implies the above matrix is invertible.

According to the polar decomposition theorem of algebra, for any invertible tensor $\underset{\sim}{F}$ there is a unique orthogonal tensor $\underset{\sim}{R}$ and unique positive definite symmetric tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ such that

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} \qquad (2.1:4)$$

or

$$\underset{\sim}{F} = \underset{\sim}{V}\underset{\sim}{R} \ . \qquad (2.1:5)$$

Geometrically this implies that any deformation from one configuration to another can be resolved uniquely into a pure stretch $\underset{\sim}{U}$ followed by a rotation $\underset{\sim}{R}$ or into a rotation $\underset{\sim}{R}$ followed by a pure stretch $\underset{\sim}{V}$. The tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are called right and left stretch tensors respectively. In order to calculate $\underset{\sim}{V}$, $\underset{\sim}{U}$, and $\underset{\sim}{R}$, we make use of the relations

$$\underset{\sim}{U}^2 = \underset{\sim}{F}^T\underset{\sim}{F} = \underset{\sim}{C} \qquad (2.1:6)$$

$$\underset{\sim}{V}^2 = \underset{\sim}{F}\underset{\sim}{F}^T = \underset{\sim}{B} \qquad (2.1:7)$$

and

$$\underset{\sim}{U} = \underset{\sim}{R}^T\underset{\sim}{V}\underset{\sim}{R} \ . \qquad (2.1:8)$$

where the superscript "T" indicates the transpose. From the above equations, we can see that the calculation of the squares of the stretch tensors is much easier than the calculation of $\underset{\sim}{U}$ and $\underset{\sim}{V}$ themselves. Hence, the tensors $\underset{\sim}{C}$ and $\underset{\sim}{B}$ have a special significance and these are called the right and left Cauchy-Green tensors respectively.

So far we have put no restrictions on the magnitudes of the deformation gradients or various other tensors defined in terms of them. Later on we will have occasion to talk about infinitesimal deformations. In order to discuss the connection between the finite and infinitesimal deformations, we define the magnitude $|\underset{\sim}{A}|$ of a tensor $\underset{\sim}{A}$ by

$$|\underset{\sim}{A}| \equiv \sqrt{\text{tr}(\underset{\sim}{A}\underset{\sim}{A}^T)} , \tag{2.1:9}$$

where tr denotes the trace.

The displacement gradient $\underset{\sim}{H}$ is defined by

$$\underset{\sim}{H} \equiv \underset{\sim}{F} - \underset{\sim}{I} . \tag{2.1:10}$$

Let us define a quantity ϵ by

$$\epsilon \equiv \sup_{\tau} |\underset{\sim}{H}(\tau)| . \tag{2.1:11}$$

We say that a deformation is infinitesimal provided that at all times $\tau < t$

$$\epsilon << 1 . \tag{2.1:12}$$

We define the (infinitesimal) strain tensor $\underset{\sim}{E}(\tau)$ and the (infinitesimal) rotation tensor $\underset{\sim}{W}(\tau)$ respectively by the equations

$$\underset{\sim}{E}(\tau) \equiv \tfrac{1}{2}[\underset{\sim}{H}(\tau) + \underset{\sim}{H}(\tau)^T] , \tag{2.1:13}$$

and

$$\underset{\sim}{W}(\tau) \equiv \tfrac{1}{2}[\underset{\sim}{H}(\tau) - \underset{\sim}{H}(\tau)^T] . \tag{2.1:14}$$

The right and left stretch tensors and the Cauchy-Green tensors are functions of τ determined by $\underset{\sim}{H}(\tau)$. We will say that a function f of τ is of the order of magnitude ϵ^n indicated by $O(\epsilon^n)$, if there exists a constant K independent of τ such that

$$|f(\tau)| < K\epsilon^n . \tag{2.1:15}$$

With this notation we would like to determine the order of magnitude of various tensors determined by $\underset{\sim}{H}(\tau)$ when $\underset{\sim}{H}(\tau) = O(\epsilon)$:

$$\underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{H} = \underset{\sim}{I} + O(\epsilon) \tag{2.1:16}$$

$$\underset{\sim}{F}^{-1} = \underset{\sim}{I} - \underset{\sim}{H} + O(\epsilon^2)$$

$$= \underset{\sim}{I} + O(\epsilon) \tag{2.1:17}$$

$$\underset{\sim}{U} = \underset{\sim}{I} + \underset{\sim}{E} + O(\epsilon^2) \tag{2.1:18}$$

$$\underset{\sim}{V} = \underset{\sim}{I} + \underset{\sim}{E} + O(\epsilon^2) \tag{2.1:19}$$

$$\underset{\sim}{C} = \underset{\sim}{I} + 2\underset{\sim}{E} + O(\epsilon^2) \tag{2.1:20}$$

$$\underset{\sim}{B} = \underset{\sim}{I} + 2\underset{\sim}{E} + O(\epsilon^2) \ . \tag{2.1:21}$$

From the equation (2.1:4) we have

$$\underset{\sim}{R} = \underset{\sim}{I} + \underset{\sim}{W} + O(\epsilon^2) \ . \tag{2.1:22}$$

It is sometimes useful to employ the present configuration as the reference configuration. We will indicate quantities computed on this basis by subscript t. Since

$$\underset{\sim}{F}(\tau) = \nabla_{\underset{\sim}{X}} \underset{\sim}{\xi}(\underset{\sim}{X}, \tau) = \nabla_{\underset{\sim}{x}} \underset{\sim}{\xi} \cdot \nabla_{\underset{\sim}{X}} \underset{\sim}{x} \ , \tag{2.1:23}$$

$$\underset{\sim}{F}(\tau) = \underset{\sim}{F}_t(\tau)\underset{\sim}{F}(t) \ . \tag{2.1:24}$$

Hence,
$$\underset{\sim}{F}_t(\tau) = \underset{\sim}{F}(\tau)\underset{\sim}{F}^{-1}(t) \tag{2.1:25}$$

and
$$\underset{\sim}{F}_t(t) = \underset{\sim}{F}(t)\ \underset{\sim}{F}^{-1}(t) = \underset{\sim}{I} \cdot \tag{2.1:26}$$

Knowing $F_t(\tau)$ we can calculate the relative right and left stretch tensors and the relative Cauchy-Green tensors. For example,

$$\underset{\sim}{C}_t(\tau) = \underset{\sim}{F}_t(\tau)^T \underset{\sim}{F}_t(\tau) \ . \tag{2.1:27}$$

Therefore,

$$\underset{\sim}{C}(\tau) = \underset{\sim}{F}(t)^T \underset{\sim}{C}_t(\tau) \underset{\sim}{F}(t) \ . \tag{2.1:28}$$

2.2 FIELD EQUATIONS OF MECHANICS

The science of mechanics is based on such notions as motion,

momentum, and stress[4]. Common experience with mechanical processes shows that certain facts are always true. These facts are formulated as general physical principles in the mathematical form of field equations. These field equations are valid for all materials. The purely mechanical processes which we now consider are governed by the following principles:

1. Conservation of mass
2. Conservation of linear momentum, and
3. Conservation of moment of momentum.

For the derivation of the field equations expressing these physical principles, the reader is referred to [14]. We will only enumerate these field equations below.

Conservation of mass:

$$\dot{\rho} + \rho \operatorname{div} \dot{\underset{\sim}{x}} = 0 \ , \tag{2.2:1}$$

where ρ is the density of the material and the superposed dot indicates material derivatives with respect to time.

Conservation of linear momentum:

$$\operatorname{div} \underset{\sim}{T} + \rho \underset{\sim}{b} = \rho \ddot{\underset{\sim}{x}} \tag{2.2:2}$$

where $\underset{\sim}{T}$ is the Cauchy stress tensor and $\underset{\sim}{b}$ is the body force per unit mass. The stress tensor $\underset{\sim}{T}$ is related to the basic concept of stress vector $\underset{\sim}{t}$ on a surface whose unit normal is $\underset{\sim}{n}$ by

$$\underset{\sim}{t}_{(n)} = \underset{\sim\sim}{T}\underset{\sim}{n} \ . \tag{2.2:2a}$$

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[4] We will consider only simple mechanical processes here. We will completely ignore considerations such as temperature, body couples, and couple stresses. We will consider electromagnetic and mechanical interactions subsequently.

Conservation of moment of momentum:

$$\underset{\sim}{T} = \underset{\sim}{T}^{T} \ . \qquad (2.2:3)$$

We must add appropriate boundary conditions to these equations. If S is the boundary of the body in some configuration, then the stress boundary conditions restrict the allowable stresses by the requirement that the applied traction vector at any point on the surface should be the same as the stress vector at that point, i.e.,

$$\underset{\sim\sim}{T}\underset{\sim}{n} = \underset{\sim}{f}(\underset{\sim}{x}) \text{ on } S \ . \qquad (2.2:4)$$

In cases where the shape of the configuration is restricted, the appropriate boundary conditions are the restrictions placed on the allowable displacements on the boundary

$$\underset{\sim}{u} = \underset{\sim}{g}(\underset{\sim}{x}) \text{ on } S \ , \qquad (2.2:5)$$

where $\underset{\sim}{u}$ is the displacement vector defined by

$$\underset{\sim}{u}(\underset{\sim}{X},t) \equiv \underset{\sim}{x}(\underset{\sim}{X},t) - \underset{\sim}{X} \ . \qquad (2.2:6)$$

In some cases, stress boundary conditions are specified on part of the boundary and displacement boundary conditions are given on the rest of the boundary.

2.3 FIELD EQUATIONS OF ELECTROMAGNETISM

The field equations of electromagnetism are derived from the following physical principles:

1. Conservation of charge, and
2. Conservation of magnetic flux.

Except on some singular surfaces these principles lead to the field equations:[5]

Conservation of charge:

$$\operatorname{div} \underset{\sim}{j} + \frac{\partial q}{\partial t} = 0 \ , \tag{2.3:1}$$

where $\underset{\sim}{j}$ is the current density and q is the charge density.

Conservation of magnetic flux:

$$\operatorname{Curl} \underset{\sim}{e} + \frac{\partial \underset{\sim}{b}}{\partial t} = 0 \tag{2.3:2}$$

and

$$\operatorname{div} \underset{\sim}{b} = 0 \ , \tag{2.3:3}$$

where $\underset{\sim}{e}$ is the electric field and $\underset{\sim}{b}$ is the magnetic flux density. The general solution of equations (2.3:1) to (2.3:3) may be expressed in terms of new fields:

$$q = \operatorname{div} \underset{\sim}{d} \ . \tag{2.3:4}$$

$$\underset{\sim}{j} = \operatorname{Curl} \underset{\sim}{h} - \frac{\partial \underset{\sim}{d}}{\partial t} \tag{2.3:5}$$

$$\underset{\sim}{b} = \operatorname{Curl} \underset{\sim}{a} \tag{2.3:6}$$

$$\underset{\sim}{e} = - \frac{\partial \underset{\sim}{A}}{\partial t} - \operatorname{grad} V \ . \tag{2.3:7}$$

Where $\underset{\sim}{d}$ is the charge potential (usually called electric displacement.)

$\underset{\sim}{h}$ is the current potential (usually called magnetic field intensity.)

$\underset{\sim}{a}$ is the magnetic potential.

V is the electric potential.

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[5] For a rigorous derivation of the field equations from the conservation principles the reader is referred to [14].

These equations must be supplemented by the Maxwell-Lorentz aether relations. The aether relations postulate that there exists fundamental constants ϵ_o and μ_o which depend only on the units of length, time, charge and magnetic flux such that the following relations are valid both within and without materials:[6]

$$\underset{\sim}{d} = \epsilon_o \underset{\sim}{e} \tag{2.3:8}$$

and

$$\underset{\sim}{h} = \frac{1}{\mu_o} \underset{\sim}{b} \, . \tag{2.3:9}$$

All the above equations are valid wherever the fields are continuous, but, across a surface of discontinuity, such as the boundary between regions of different material properties, the fields suffer jumps. In order to relate the fields on both sides, consider a surface of discontinuity with unit-normal $\underset{\sim}{\nu}$, moving with speed u_n. Then the following jump conditions hold:[7]

$$\underset{\sim}{\nu} \times [\![\underset{\sim}{e}]\!] - u_n [\![\underset{\sim}{b}]\!] = 0 \, , \tag{2.3:10}$$

$$[\![\underset{\sim}{b}]\!] \cdot \underset{\sim}{\nu} = 0 \, , \tag{2.3:11}$$

and

$$[\![\underset{\sim}{j}]\!] \cdot \underset{\sim}{\nu} - u_n [\![q]\!] = 0 \tag{2.3:12}$$

where the symbol $[\![\ ]\!]$ indicates the jump in the quantity across the surface of discontinuity.

It is customary to consider the total charge to be made up of two parts:

1. q_b the bound charge, and
2. q_f the free charge.

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[6] We follow the point of view of Toupin [14].

[7] For a derivation of these jump conditions, the reader is referred to [14].

Then

$$q = q_b + q_f \ . \qquad (2.3:13)$$

The total current is considered to be the sum of three parts:

1. $\underset{\sim}{j}_f$ the free current,
2. $\underset{\sim}{j}_p$ the polarization current, and
3. $\underset{\sim}{j}_m$ the magnetization current;

then

$$\underset{\sim}{j} = \underset{\sim}{j}_f + \underset{\sim}{j}_p + \underset{\sim}{j}_m \ . \qquad (2.3:14)$$

The auxiliary charge and currents are defined in terms of the polarization $\underset{\sim}{p}$ and magnetization $\underset{\sim}{m}$:

$$q_b = -\text{div}\ \underset{\sim}{p} \ , \qquad (2.3:15)$$

$$\underset{\sim}{j}_p = \frac{\partial \underset{\sim}{p}}{\partial t} + \text{Curl}(\underset{\sim}{p} \times \underset{\sim}{v}) \ , \qquad (2.3:16)$$

and

$$\underset{\sim}{j}_m = \text{Curl}\ \underset{\sim}{m} \ , \qquad (2.3:17)$$

where $\underset{\sim}{v}$ is the velocity of the medium.

Auxiliary potentials $\underset{\sim}{h}$ and $\underset{\sim}{d}$, which we call partial potentials, are defined by

$$\underset{\sim}{h} = \frac{\underset{\approx}{1}\underset{\sim}{b}}{N_o} - \underset{\sim}{m} - \underset{\sim}{p} \times \underset{\sim}{v} \qquad (2.3:18)$$

and

$$\underset{\sim}{d} = \epsilon_o \underset{\sim}{e} + \underset{\sim}{p} \ . \qquad (2.3:19)$$

In terms of the partial potentials, we have

$$\underset{\sim}{j}_f = \text{Curl}\ \underset{\sim}{h} - \frac{\partial \underset{\sim}{d}}{\partial t} \ , \qquad (2.3:20)$$

and

$$q_f = \text{div}\ \underset{\sim}{d} \ . \qquad (2.3:21)$$

The jump conditions for the partial potentials across a surface of discontinuity, with normal $\underset{\sim}{\nu}$ and apeed u_n are

$$[\![\underset{\sim}{d}]\!] \cdot \underset{\sim}{\nu} = 0 \tag{2.3:22}$$

and

$$\underset{\sim}{\nu} \times [\![\underset{\sim}{h}]\!] + u_n [\![\underset{\sim}{d}]\!] = 0 \, . \tag{2.3:23}$$

2.4 MATERIAL INDIFFERENCE

The field equations, when combined with the jump conditions, form an under determined system of equations. This can be expected because the equations are valid for all materials and fail to account for the characteristics of the particular material under study. To make the system of equations determinate, we introduce additional equations defining ideal materials. These additional equations are called constitutive equations.

Constitutive equations are often formulated as functional relations between various state variables. These functionals cannot be arbitrarily chosen, but must satisfy some broadly established general principles.[8]

The principle variously known as the principle of material indifference, principle of material objectivity, or principle of isotropy of space is one such physical principle. It is a statement of the requirement that "the response of the material is independent of the observer."

Even though implicit use of this principle can be found in earlier literature (the reader can find some historical remarks

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[8] For a discussion of these principles the reader is referred to [14].

about this principle in [15]), the first satisfactory statement is due to Noll [16] under the name "Principle of Isotropy of Space." The following statement of this principle was subsequently given by Truesdell and Noll [15]:

Constitutive equations must be invariant under changes of frame of reference. If a constitutive equation is satisfied for a process consisting of a motion and symmetric stress tensor given by

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t) \ , \tag{2.4:1}$$

$$\underset{\sim}{T} = \underset{\sim}{T}(\underset{\sim}{X},t) \tag{2.4:2}$$

then it must be satisfied also for any equivalent process $\{x^*,T^*\}$ given by

$$\underset{\sim}{x}^* = \underset{\sim}{x}^*(\underset{\sim}{X},t^*) = \underset{\sim}{Q}(t)\underset{\sim}{x}(\underset{\sim}{X},t) + \underset{\sim}{c}(t) \ , \tag{2.4:3}$$

$$\underset{\sim}{T}^* = \underset{\sim}{T}^*(\underset{\sim}{X},t^*) = \underset{\sim}{Q}(t)\underset{\sim}{T}(\underset{\sim}{X},t)\underset{\sim}{Q}(t)^T \ , \tag{2.4:4}$$

$$t^* = t-a \tag{2.4:5}$$

where $\underset{\sim}{c}(t)$ is an arbitrary point, $\underset{\sim}{Q}(t)$ an arbitrary orthogonal tensor valued function of time, and "a" an arbitrary number.

The transformation $\underset{\sim}{Q}(t)$ represents a time dependent rotation of the coordinate frame, and $\underset{\sim}{c}(t)$ a time dependent shift of the origin of the frame. As an example, consider a typical mechanical constitutive equation of the form

$$\underset{\sim}{T}(\underset{\sim}{X},t) = \mathfrak{F}_t\{\underset{\sim}{x}(\underset{\sim}{X},{}^{t}_{-\infty}\tau)\} \tag{2.4:6}$$

where $\mathfrak{F}_t$ is a functional and the subscript t representing that the functional relation is dependent on t. Then, according to the principle of material indifference, the functional $\mathfrak{F}_t$ must be such that

$$\underset{\sim}{T}^*(\underset{\sim}{X},t^*) = \mathfrak{F}_{t^*}\{\underset{\sim}{x}^*(\underset{\sim}{X},\overset{t^*}{\underset{-\infty}{\tau}})\} \qquad (2.4:7)$$

under the group of transformations given by (2.4:3) to (2.4:5). Equation (2.4:7) may also be viewed as stating that constitutive functional relations should be the same in two reference frames which have a relative motion with respect to one another.

In subsequent chapters, we will have occasion to use the various kinematical quantities in the starred reference frame. We will indicate these quantities by a star as a superscript. Using ordinary differentiation rules, various kinematical quantities in the starred reference frame are calculated below. Deformation gradient $\underset{\sim}{F}$:

$$\begin{aligned}\underset{\sim}{F}^*(t) &= \nabla_{\underset{\sim}{X}}\underset{\sim}{x}^*(\underset{\sim}{X},t)\\ &= \underset{\sim}{Q}(t)\nabla_{\underset{\sim}{X}}\underset{\sim}{x}(\underset{\sim}{X},t)\\ &= \underset{\sim}{Q}(t)\underset{\sim}{F}(t)\end{aligned} \qquad (2.4:8)$$

Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{B}$:

$$\begin{aligned}\underset{\sim}{C}^*(t) &= \underset{\sim}{F}^*(t)^T\underset{\sim}{F}^*(t)\\ &= \underset{\sim}{F}(t)^T\underset{\sim}{Q}(t)^T\underset{\sim}{Q}(t)\underset{\sim}{F}(t)\\ &= \underset{\sim}{F}(t)^T\underset{\sim}{F}(t)\\ &= \underset{\sim}{C}(t)\end{aligned} \qquad (2.4:9)$$

$\underset{\sim}{C}(t)$ is invariant under the transformation (2.4:3)

$$\begin{aligned}\underset{\sim}{B}^*(t) &= \underset{\sim}{F}^*(t)\underset{\sim}{F}^*(t)^T\\ &= \underset{\sim}{Q}(t)\underset{\sim}{F}(t)\underset{\sim}{F}(t)^T\underset{\sim}{Q}(t)^T\\ &= \underset{\sim}{Q}(t)\underset{\sim}{B}(t)\underset{\sim}{Q}(t)^T\end{aligned} \qquad (2.4:10)$$

2.5 ISOTROPY

In mechanics of deformable bodies, the adjective "isotropic" is often used very loosely. It is a common practice to say "a material is isotropic if it does not have any preferred directions." This definition is vague. We will make the concept of isotropy precise [16].

We define a material to be isotropic if and only if the constitutive relation of the material is invariant under the transformation of reference configuration:

$$\underset{\sim}{X} \longrightarrow \underset{\sim}{X}^{**} = \underset{\sim}{P}^{T}\underset{\sim}{X} + \underset{\sim}{a} \qquad (2.5:1)$$

$$\underset{\sim}{T} \longrightarrow \underset{\sim}{T}^{**} = \underset{\sim}{T} \qquad (2.5:2)$$

where $\underset{\sim}{P}$ is an arbitrary proper orthogonal tensor function of time and $\underset{\sim}{a}(t)$ an arbitrary vector. A material which is not isotropic is called anisotropic.

This definition has one drawback: the use of the phrase "the constitutive relation of the material." An example will make this clear.

Consider the following possible problems involving the same material:

1. Evaluation of the stress within the material due to a given applied loading, and
2. Passage of electromagnetic waves through a dielectric.

The first problem is governed by the field equations of mechanics and the second problem is governed by the equations of electromagnetism. In both cases we should add appropriate con-

stitutive equations in order to solve the problem. In the first problem, the constitutive equation relating the stress and strain tensor is the one we should add, whereas in the second case the equation relating the electric field and electric displacement is the appropriate one. Hence, the phrase "the constitutive relation of the material" is meaningful only when the problem under consideration is specified.

This drawback in definition can be avoided by defining mechanical isotropy, electromagnetic isotropy, etc., separately. For example, we define a material to be mechanically isotropic if and only if the mechanical constitutive relation of the material is invariant under the transformation (2.5:1). Whenever the situation does not make it clear, we will specify the type of isotropy under consideration. A material may be isotropic in certain respects and anisotropic in other respects. For example, mechanical isotropy does not necessarily imply electromagnetic isotropy of a material. In later sections we will have occasion to talk about such materials.

The various kinematical quantities under the transformation of the reference configuration given by equation (2.5:1) can be easily calculated using ordinary rules of differentiation. Following are the calculations of some of these kinematical quantities, where the superscript "double star" is used to indicate them.

Deformation gradient $\underset{\sim}{F}$:

$$\underset{\sim}{F}^{**}(t) = \nabla_{\underset{\sim}{X}}{}^{**}\, \underset{\sim}{x}(\underset{\sim}{X},t)$$

$$= (\nabla_{\underset{\sim}{X}}\underset{\sim}{x})\nabla_{\underset{\sim}{X}}{}^{**}\, \underset{\sim}{X}$$

$$= \underset{\sim}{F}(t)\underset{\sim}{P}(t)\cdot \qquad (2.5:3)$$

Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{B}$:

$$\underset{\sim}{C}^{**}(t) = \underset{\sim}{F}^{**}(t)^{T}\underset{\sim}{F}^{**}(t)$$

$$= \underset{\sim}{P}(t)^{T}\underset{\sim}{F}(t)^{T}\underset{\sim}{F}(t)\underset{\sim}{P}(t)$$

$$= \underset{\sim}{P}(t)^{T}\underset{\sim}{C}(t)\underset{\sim}{P}(t) \qquad (2.5:4)$$

$$\underset{\sim}{B}^{**}(t) = \underset{\sim}{F}^{**}(t)\underset{\sim}{F}^{**}(t)^{T}$$

$$= \underset{\sim}{F}(t)\underset{\sim}{P}(t)\underset{\sim}{P}(t)^{T}\underset{\sim}{F}(t)^{T}$$

$$= \underset{\sim}{B}(t)\cdot \qquad (2.5:5)$$

$\underset{\sim}{B}(t)$ is invariant under the transformation (2.5:1).

Left Stretch tensor $\underset{\sim}{V}$:

$$\underset{\sim}{V}^{**}(t) = [\underset{\sim}{B}^{**}(t)]^{\frac{1}{2}}$$

$$= \underset{\sim}{B}(t)^{\frac{1}{2}}$$

$$= \underset{\sim}{V}(t)\cdot \qquad (2.5:5)$$

$\underset{\sim}{V}(t)$ is invariant under transformation (2.5:1)

Rotation tensor $\underset{\sim}{R}$: From equation (2.1:5), we have

$$\underset{\sim}{V}(t) = \underset{\sim}{F}(t)\underset{\sim}{R}(t)^{T}\cdot \qquad (2.5:6)$$

Hence, from equation (2.5:5), we can write

$$\underset{\sim}{F}(t)\underset{\sim}{R}(t)^{T} = \underset{\sim}{F}^{**}(t)\underset{\sim}{R}^{**}(t)^{T}$$

$$= \underset{\sim}{F}(t)\underset{\sim}{P}(t)\underset{\sim}{R}^{**}(t)^{T}$$

or

$$\underset{\sim}{R}^{**}(t)^T = \underset{\sim}{P}(t)^T \underset{\sim}{R}(t)^T. \qquad (2.5:7)$$

Taking the transpose of both sides

$$\underset{\sim}{R}^{**}(t) = \underset{\sim}{R}(t)\underset{\sim}{P}(t). \qquad (2.5:8)$$

3. CONSTITUTIVE EQUATIONS OF A SIMPLE MATERIAL

3.1 GENERAL REMARKS

In photomechanics we have to consider both mechanical and electromagnetic fields at the same time. For such cases, not one but a set of constitutive equations relating the various field quantities are necessary for an exact theory. One such set of very general constitutive equations might be

$$\underset{\sim}{T} = \underset{\tau=-\infty}{\overset{t}{\underset{\sim}{\mathfrak{I}}}} \{\underset{\sim}{x}(\underset{\sim}{X},\tau),\underset{\sim}{e}(\tau),\underset{\sim}{b}(\tau)\}, \qquad (3.1:1)$$

$$\underset{\sim}{p} = \underset{\tau=-\infty}{\overset{t}{\underset{\sim}{\mathfrak{P}}}} \{\underset{\sim}{x}(\underset{\sim}{X},\tau),\underset{\sim}{e}(\tau),\underset{\sim}{b}(\tau)\}, \qquad (3.1:2)$$

$$\underset{\sim}{m} = \underset{\tau=-\infty}{\overset{t}{\underset{\sim}{\mathfrak{M}}}} \{\underset{\sim}{x}(\underset{\sim}{X},\tau),\underset{\sim}{e}(\tau),\underset{\sim}{b}(\tau)\}, \qquad (3.1:3)$$

$$\underset{\sim}{j} = \underset{\tau=-\infty}{\overset{t}{\underset{\sim}{\mathfrak{X}}}} \{\underset{\sim}{x}(\underset{\sim}{X},\tau),\underset{\sim}{e}(\tau),\underset{\sim}{b}(\tau)\}. \qquad (3.1:4)$$

We shall confine our attention to ideal dielectrics defined by the relations $\underset{\sim}{m} = o$, $\underset{\sim}{j}_f = o$, $\underset{\sim}{p}$ depends on the present value of $\underset{\sim}{e}$ and does not depend on $\underset{\sim}{b}$. Furthermore, we wish to consider only problems involving very weak electromagnetic fields (propagation of light). Therefore, we can neglect the dependence of $\underset{\sim}{T}$ on $\underset{\sim}{e}$ and $\underset{\sim}{b}$ in (3.1:1), and take $\underset{\sim}{p}$ as linearly dependent on $\underset{\sim}{e}$:

$$\underset{\sim}{T} = \underset{\sim}{\mathfrak{I}}\{\underset{\sim}{x}(\underset{\sim}{X},\tau)\} , \qquad (3.1:5)$$

$$\underset{\sim}{p} = \underset{\sim}{\mathfrak{P}}\{\underset{\sim}{x}(\underset{\sim}{X},\tau)\}\underset{\sim}{e} . \qquad (3.1:6)$$

The constitutive relation (3.1:5) will be considered in Section 3.2 and the relation (3.1:6) in Section 3.3.

3.2 MECHANICAL CONSTITUTIVE EQUATION OF A SIMPLE MATERIAL

In this section we will discuss mechanical constitutive equations under the following two requirements:

1. The principle of material indifference must be satisfied, and
2. The material is mechanically isotropic.

We will restrict ourselves to the purely mechanical case and neglect the influence of any other fields such as electromagnetic fields.

A simple material is defined as one, in which the stress at any point $\underset{\sim}{X}$ at the present time t depends only on the history of the deformation gradient $\underset{\sim}{F}(\tau)$ in the neighborhood of the point $\underset{\sim}{X}$ under consideration:

$$\underset{\sim}{T}(\underset{\sim}{X},t) = \underset{\sim}{\mathfrak{I}}\{\underset{\sim}{F}(\underset{-\infty}{\overset{t}{\tau}})\} . \qquad (3.2:1)$$

The functional in equation (3.2:1) cannot be arbitrary. It is restricted to the class of functionals that satisfy the principle of material indifference. According to this principle, the constitutive equation (3.1:1) should have the same form in the new frame of coordinates:

$$\underset{\sim}{T}^*(\underset{\sim}{X},t^*) = \underset{\sim}{\mathfrak{I}}\{\underset{\sim}{F}^*(\underset{-\infty}{\overset{t^*}{\tau}})\} . \qquad (3.2:2)$$

Making use of equations (2.4:4), (2.4:8), and (3.2:1) leads to the relation

$$\underset{\sim}{Q}(t)\mathcal{J}\{\underset{\sim}{F}(\tau)\}\underset{\sim}{Q}^{T}(t) = \mathcal{J}\{\underset{\sim}{Q}(\tau)\underset{\sim}{F}(\tau)\} \tag{3.2:3}$$

which must be identically satisfied for every orthogonal tensor $\underset{\sim}{Q}(t)$ and every invertible tensor $\underset{\sim}{F}(t)$.

If the material is mechanically isotropic, this puts further restrictions on the constitutive relation (3.2:1). The mechanical constitutive equation (3.2:1) must then remain invariant under the transformation (2.5:1):

$$\underset{\sim}{T}(\underset{\sim}{X}^{**},t) = \mathcal{J}\{\underset{\sim}{F}^{**}(\underset{\sim}{X}^{**},\underset{-\infty}{\overset{t}{\tau}})\} , \tag{3.2:4}$$

where the double star indicates quantities under transformation (2.5:1). Using equations (2.5:2) and (2.5:3), we obtain the relation

$$\mathcal{J}\{\underset{\sim}{F}(\underset{\sim}{X},\tau)\} = \mathcal{J}\{\underset{\sim}{F}(\underset{\sim}{X},\tau)\underset{\sim}{P}(\tau)\} \tag{3.2:5}$$

which must be satisfied identically for every proper orthogonal tensor $\underset{\sim}{P}$ and every invertible tensor $\underset{\sim}{F}$.

Equations (3.2:3) and (3.2:5) express the restrictions imposed on the mechanical constitutive equation by the principle of material indifference and material isotropy respectively. Explicit representation for the constitutive functionals subject to either of these restrictions can be obtained by making use of the right and left deformation tensors. We will consider the use of these tensors separately. For the sake of simplicity in writing, we omit indication of the dependence of various quantities on time.

a. Use of Right Deformation Tensor

The mechanical constitutive equation satisfying the restriction (3.2:3) imposed by the principle of material indifference can be obtained by using the right deformation tensor. Making use of equation (2.1:4), equation (3.2:3) can be written as follows:

$$\underset{\sim}{Q}\underset{\sim}{\mathcal{F}}\{\underset{\sim}{R}\underset{\sim}{U}\}\underset{\sim}{Q}^{T} = \underset{\sim}{\mathcal{F}}\{\underset{\sim}{Q}\underset{\sim}{R}\underset{\sim}{U}\} \qquad (3.2:7)$$

for every orthogonal transformation $\underset{\sim}{Q}$. Let us take $\underset{\sim}{Q}$ to be equal to $\underset{\sim}{R}^{T}$. With this choice of $\underset{\sim}{Q}$, equation (3.2:7) takes the form

$$\underset{\sim}{T} = \underset{\sim}{\mathcal{F}}\{\underset{\sim}{F}\} = \underset{\sim}{R}\underset{\sim}{\mathcal{F}}\{\underset{\sim}{U}\}\underset{\sim}{R}^{T}. \qquad (3.2:8)$$

It can be seen by direct substitution that this restriction on $\underset{\sim}{\mathcal{F}}$ is also sufficient. Equation (3.2:8) represents the general form of the mechanical constitutive equation of a simple material satisfying the principle of material indifference. It is equally valid for both isotropic and anisotropic materials. If the material is also isotropic, then equation (3.2:8) must be invariant under the transformation (2.5:1):

$$\underset{\sim}{T}^{**} = \underset{\sim}{R}^{**}\underset{\sim}{\mathcal{F}}\{\underset{\sim}{U}^{**}\}\underset{\sim}{R}^{**T} . \qquad (3.2:9)$$

Making use of the equations (2.5:2), (2.5:3) and (2.5:4), this equation can be reduced to

$$\underset{\sim}{P}^{T}\underset{\sim}{\mathcal{F}}\{\underset{\sim}{U}\}\underset{\sim}{P} = \underset{\sim}{\mathcal{F}}\{\underset{\sim}{P}^{T}\underset{\sim}{U}\underset{\sim}{P}\} . \qquad (3.2:10)$$

Functionals satisfying this type of relation are called isotropic functionals.[9] Thus, if the material is isotropic, the constitutive functional $\underset{\sim}{\mathcal{F}}$ must be an isotropic functional.

[9] For an exact definition of isotropic function see Truesdell and Noll [15], pp 22-23.

b. Use of Left Deformation Tensor

The general mechanical constitutive equation satisfying the restriction (3.2:3) imposed by the mechanical isotropy of the material can be obtained in a simple form by using the left deformation tensor. Use of equation (2.1:5) in equation (3.2:5) leads to

$$\mathcal{G}\{\underset{\sim}{F}\} = \mathcal{G}\{\underset{\sim}{V}\underset{\sim}{R}\underset{\sim}{P}\} , \tag{3.2:11}$$

for every proper orthogonal transformation $\underset{\sim}{P}$. Let us choose $\underset{\sim}{P}$ equal to $\underset{\sim}{R}^T$. With this choice, equation (3.2:11) leads to

$$\underset{\sim}{T} = \mathcal{G}\{\underset{\sim}{F}\} = \mathcal{G}\{\underset{\sim}{V}\} = \tilde{\mathcal{G}}(\underset{\sim}{B}) . \tag{3.2:12}$$

This equation, in order to represent a valid constitutive equation, must satisfy the principle of material indifference. It must be invariant under transformation (2.4:3) - (2.4:5):

$$\underset{\sim}{T}^* = \tilde{\mathcal{G}}\{\underset{\sim}{B}^*\} . \tag{3.2:13}$$

From equations (2.4:4) and (2.4:10), we get

$$\underset{\sim}{Q}\tilde{\mathcal{G}}\{\underset{\sim}{B}\}\underset{\sim}{Q}^T = \tilde{\mathcal{G}}\{\underset{\sim}{Q}\underset{\sim}{B}\underset{\sim}{Q}^T\} . \tag{3.2:14}$$

That is, $\tilde{\mathcal{G}}$ must be an isotropic functional.

Summarizing the results of this Section, we have the following statement:

The mechanical constitutive equation of an isotropic simple material has one of the forms of equations (3.2:8) or (3.2:12) where the functionals $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are isotropic functionals.

3.3 PHOTOMECHANICAL CONSTITUTIVE EQUATIONS

We consider an ideal dielectric defined by the relation (3.1:6)

$$\underset{\sim}{p} = \underset{\sim}{P}\{\underset{\sim}{x}(\underset{\sim}{X},\underset{-\infty}{\overset{t}{\tau}})\}\underset{\sim}{e} \tag{3.3:1}$$

From (2.3:19), we can define a new tensor $\underset{\sim}{K}$ such that

$$\underset{\sim}{d} = \epsilon_o \underset{\sim}{K}\underset{\sim}{e} \tag{3.3:2}$$

The tensor $\underset{\sim}{K}$ is called the dielectric tensor.

We consider only the class of materials for which the dielectric tensor at any particle X and time t is completely determined by the history of the deformation gradient:

$$\underset{\sim}{K}(\underset{\sim}{X},t) = \underset{\sim}{\mathfrak{K}}\{\underset{\sim}{F}(\underset{-\infty}{\overset{t}{\tau}})\} \ . \tag{3.3:3}$$

One might also consider the following possible constitutive equations:

$$\underset{\sim}{K}(\underset{\sim}{X},t) = \underset{\sim}{\mathfrak{K}}_1\{\underset{\sim}{T}(\underset{-\infty}{\overset{t}{\tau}})\} \ , \tag{3.3:4}$$

or

$$\underset{\sim}{K}(\underset{\sim}{X},t) = \underset{\sim}{\mathfrak{K}}_2\{\underset{\sim}{T}(\underset{-\infty}{\overset{t}{\tau}}),\underset{\sim}{F}(\underset{-\infty}{\overset{t}{\tau}})\} \ . \tag{3.3:5}$$

Since we assume that all the materials under consideration are mechanically simple, use of the mechanical constitutive equation (3.2:1) reduces equations (3.3:4) and (3.3:5) to the form of equation (3.3:3). Hence, in this section we consider only equation (3.3:3).

We make the following assumptions:

1. The electromagnetic quantities $\underset{\sim}{d}$ and $\underset{\sim}{e}$ transform like vectors from one frame of spacial coordinates to another.

2. The principle of material indifference may be extended to

assert that the constitutive equation (3.3:3) must be invariant under the transformation (2.4:3), and

3. The material is photomechanically isotropic so that the constitutive relation (3.3:3) is invariant under the transformation (2.5:1).

In order to determine transformation of $\underset{\sim}{K}$ under (2.4:3), let us see how the equation (3.3:2) transforms under the change of spacial frame given by equation (2.4:3)

$$\underset{\sim}{d}^* = \epsilon_o \underset{\sim}{K}^* \underset{\sim}{e}^* . \tag{3.3:6}$$

But from the first assumption,

$$\underset{\sim}{d}^* = \underset{\sim\sim}{Q}\underset{\sim}{d} \tag{3.3:7}$$

and

$$\underset{\sim}{e}^* = \underset{\sim\sim}{Q}\underset{\sim}{e} . \tag{3.3:8}$$

Hence

$$\begin{aligned} \underset{\sim\sim}{Q}\underset{\sim}{d} &= \underset{\sim\sim}{Q}\epsilon_o \underset{\sim\sim}{K}\underset{\sim}{e} \\ &= \epsilon_o \underset{\sim}{K}^* \underset{\sim\sim}{Q}\underset{\sim}{e} \end{aligned} \tag{3.3:9}$$

or

$$(\underset{\sim\sim}{Q}\underset{\sim\sim}{K} - \underset{\sim}{K}^*\underset{\sim\sim}{Q})\underset{\sim}{e} = 0 . \tag{3.3:10}$$

Equation (3.3:9) must be satisfied for all $\underset{\sim}{e}$. Thus,

$$\underset{\sim}{K}^* = \underset{\sim\sim}{Q}\underset{\sim\sim}{K}\underset{\sim\sim}{Q}^T . \tag{3.3:11}$$

The dielectric tensor $\underset{\sim}{K}$, analogous to $\underset{\sim}{T}$, transforms like a second order tensor under (2.4:3). Hence, we can use exactly the same reasoning as was used in Section 3.2 to show that the requirement of material indifference and isotropy reduces equation (3.3:3) to either one of the forms.

$$\underset{\sim}{K} = \underset{\sim}{R}\underset{\sim}{\mathcal{R}}\{\underset{\sim}{C}\}\underset{\sim}{R}^T \qquad (3.3:12)$$

or

$$\underset{\sim}{K} = \underset{\sim}{\mathcal{R}}\{\underset{\sim}{B}\}; \qquad (3.3:13)$$

Where the functional must be isotropic:

$$\underset{\sim}{Q}\underset{\sim}{\mathcal{R}}\{\underset{\sim}{C}\}\underset{\sim}{Q}^T = \underset{\sim}{\mathcal{R}}\{\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{Q}^T\} \ . \qquad (3.3:14)$$

4. APPROXIMATE MECHANICAL CONSTITUTIVE EQUATIONS

4.1 FADING MEMORY

So far we have dealt only with the general mechanical constitutive relation. In this chapter we will investigate some approximations of practical value. For a more thorough treatment of the subject matter of this chapter, the reader is referred to References (14) and (15). We will give only the essentials, following the above references.

In Section 3.2 we have defined a simple material as one in which the stress at the present time is determined by the entire history of the deformation gradient. We now consider materials, known as materials with fading memory, for which deformations that occurred in the distant past have less influence in determining the present stress than those that occurred in the recent past.

In order to give a precise mathematical formulation to the concept of fading memory, let us introduce the influence function (Coleman and Noll) or obliviator (Truesdell and Noll).

A function h is called an obliviator of order n>0 if it satisfies the following conditions.

a) $h(s)$ is defined for $0<s<\infty$ and has positive real values: $h(s)>0$

b) $h(s)$ is normalized by the condition

$$h(0) = 1 \tag{4.1:1}$$

c) $h(s)$ decays to zero in such a way that

$$\lim_{s\to\infty} s^n h(s) = 0 \tag{4.1:2}$$

monotonically for large s.

For example,

$$h(s) = \frac{1}{(s+1)^p} \tag{4.1:3}$$

is an obliviator of order $n<p$. An exponential is an

$$h(s) = e^{-\beta s}, \quad \beta>0 , \tag{4.1:4}$$

obliviator of arbitrary order.

Consider the linear function space of the histories of symmetric tensors $\underset{\sim}{G}(s)$. The parameter s is to be regarded as time measured backward from the present. We define the recollection of a history $\underset{\sim}{G}(s)$ as the norm of history $\underset{\sim}{G}(s)$ defined by

$$\|\underset{\sim}{G}(s)\|_h = \left(\int_0^\infty [h(s)|\underset{\sim}{G}(s)|]^2 ds\right)^{\frac{1}{2}} , \tag{4.1:5}$$

where

$$|\underset{\sim}{G}(s)| = \sqrt{tr[\underset{\sim}{G}(s)^2]} \tag{4.1:6}$$

is the magnitude of the tensor $\underset{\sim}{G}(s)$. The collection of all histories with finite recollection forms a Hilbert Space H. The inner product of two histories $\underset{\sim}{G}(s)$ and $\underset{\sim}{H}(s)$ in H is given by

$$\langle \underset{\sim}{G}(s), \underset{\sim}{H}(s)\rangle_h = \int_0^\infty Tr[\underset{\sim}{G}(s)\cdot\underset{\sim}{H}(s)]h(s)^2 ds . \tag{4.1:7}$$

Now the principle of fading memory can be put in the following

precise mathematical form[10]: a functional $\mathfrak{F}$ is a fading memory functional if there exists an obliviator of order greater than $n + 1/2$ such that the response functional $\mathfrak{F}\{\underset{\sim}{G}\ (\overset{\infty}{\underset{0}{s}})\}$ is defined and n times Frechet-differentiable in a neighborhood of the zero history of the function space H.

It is important to note here that the above principle does not require any admissible history $\underset{\sim}{G}(s)$ to be continuous. Hence this principle can be applied to histories of the type that occur in stress-relaxation experiments.

We can see the implications of the principle of the fading memory, from the way we defined the norm in the space of histories. As the integral defining the norm is weighted with a decaying function, two histories can be close to one another if the values of $\underset{\sim}{G}(s)$ are close enough in recent past (small values of s) even if they are far apart in distant past (large values of s).

4.2 FINITE LINEAR VISCOELASTICITY

In this section we will derive the constitutive relation defining finite linear viscoelasticity. This theory is due to Coleman and Noll (13). This section follows their presentation.

We have shown in Section 3.2 that the general constitutive relation of a simple material (restricted by the principle of material indifference) has the form (3.2:8):

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[10] Truesdell and Noll (15), call this the stronger principle of fading memory.

$$\underset{\sim}{R}(t)^T\underset{\sim}{T}(t)\underset{\sim}{R}(t) = \underset{\sim}{\mathcal{J}}\{\underset{\sim}{U}(t-\overset{\infty}{\underset{0}{s}})\} . \qquad (4.2:1)$$

The stretch tensor $\underset{\sim}{U}(\tau)$ is computed with the initial state as the reference state. However, for the puspose of the present chapter, we would like the arguments of the response functional $\underset{\sim}{\mathcal{J}}$ to be quantities computed with the present state as reference configuration.

From (2.1:6), we see that $\underset{\sim}{U}(\tau)$ is determined by $\underset{\sim}{C}(\tau)$. From (2.1:4) and (2.1:28), we find

$$\underset{\sim}{C}(\tau) = \underset{\sim}{U}(t)^T\hat{\underset{\sim}{C}}_t(\tau)\underset{\sim}{U}(t) \qquad (4.2:2)$$

where

$$\hat{\underset{\sim}{C}}_t(\tau) = \underset{\sim}{R}(t)^T\underset{\sim}{C}_t(\tau)\underset{\sim}{R}(t) . \qquad (4.2:3)$$

Therefore, $\underset{\sim}{C}(\tau)$ is determined by $\hat{\underset{\sim}{C}}_t(\tau)$ and $\underset{\sim}{C}(t)$. This enables us to derive a new functional $\underset{\sim}{\mathcal{R}}$ such that

$$\hat{\underset{\sim}{T}}(t) = \underset{\sim}{\mathcal{R}}\{\underset{\sim}{G}_t(t-\overset{\infty}{\underset{0}{s}});\underset{\sim}{C}(t)\} \qquad (4.2:4)$$

where

$$\underset{\sim}{G}_t(\tau) = \hat{\underset{\sim}{C}}_t(\tau)-\underset{\sim}{I} , \qquad (4.2:5)$$

$$\hat{\underset{\sim}{T}}(t) = \underset{\sim}{R}(t)^T\underset{\sim}{T}(t)\underset{\sim}{R}(t) , \qquad (4.2:6)$$

and $\underset{\sim}{\mathcal{R}}$ depends on the history $\underset{\sim}{G}_t(\tau)$ but only on the present value of $\underset{\sim}{C}$.

If the material has always been at rest

$$\hat{\underset{\sim}{C}}_t(t-s) = \underset{\sim}{I}$$

and hence the right side becomes a function of $\underset{\sim}{C}(t)$ only. This function $\underset{\sim}{f}(\underset{\sim}{C}(t))$ is called the "equilibrium term." Thus, equation (4.2:4) may be written in the alternate form

$$\hat{\underset{\sim}{T}}(t) = \underset{\sim}{f}(\underset{\sim}{C}(t)) + \underset{\sim}{\mathfrak{R}}\{\underset{\sim}{G}_t(t-\overset{\infty}{\underset{s=\infty}{s}});\underset{\sim}{C}(t)\} \qquad (4.2:7)$$

where the new functional $\underset{\sim}{\mathfrak{R}}$, denoted by the same symbol, is such that

$$\underset{\sim}{\mathfrak{R}}\{0;\underset{\sim}{C}(t)\} = 0 \ . \qquad (4.2:8)$$

The functional $\underset{\sim}{\mathfrak{R}}$ can be approximated by a bounded, homogeneous, polynomial functional of $\underset{\sim}{G}_t(t-s)$ under suitable conditions. In order to achieve this, Coleman and Noll (13) assumed the following additional smoothness requirements:

a) The Frechet differentiability of the response functional postulated in the principle of fading memory is uniform in the tensor parameter $\underset{\sim}{C}(t)$, and

b) The tensor function $\underset{\sim}{f}(\underset{\sim}{C})$ of equation (4.2:7) is n times continuously differentiable.

These two assumptions together with the principle of fading memory are sufficient to justify the approximation[11]

$$\hat{\underset{\sim}{T}} = \underset{\sim}{f}(\underset{\sim}{C}) + \sum_{k=1}^{n} \frac{1}{k!}\delta^k\underset{\sim}{\mathfrak{R}}\{\underset{\sim}{G}_t(s);\underset{\sim}{C}\} + o\left(\|\underset{\sim}{G}_t(s)\|_h^n\right) , \qquad (4.2:9)$$

where the order symbol "o" is used in the sense

$$\lim_{\epsilon \to o} \frac{o(\epsilon)}{\epsilon} = 0 \qquad (4.2:10)$$

[11] For convenience in writing the explicit dependence of all the quantities on the present time t is suppressed in most of the equations hereafter: i.e., $\underset{\sim}{T} \equiv \underset{\sim}{T}(t)$, etc.

and $\delta^k \underset{\sim}{\mathfrak{R}}\{\ \}$ is the k^{th} variation of the functional $\underset{\sim}{\mathfrak{R}}\{\ \}$.

If we consider the particular case of n = 1, we obtain

$$\hat{\underset{\sim}{T}} = \underset{\sim}{f}(\underset{\sim}{C}) + \delta\underset{\sim}{\mathfrak{R}}\{\underset{\sim}{G}_t(s);\underset{\sim}{C}\} + o\left(\|\underset{\sim}{G}_t(s)\|_h^2\right). \qquad (4.2:11)$$

Consider deformation histories $\underset{\sim}{G}_t(s)$ which are small in the recent past. Because the integral defining the norm is weighted with an obliviator, the norm of $\underset{\sim}{G}_t(s)$ can be considered small even if the deformations are large in distant past. For such deformations the last term in equation (4.2:11) can be neglected compared to the other terms and we have the approximation

$$\underset{\sim}{R}^T\underset{\sim}{T}\underset{\sim}{R} \simeq \underset{\sim}{f}(\underset{\sim}{C}) + \delta\underset{\sim}{\mathfrak{R}}\{\underset{\sim}{G}_t(s);\underset{\sim}{C}\}. \qquad (4.2:12)$$

Now, let us make use of a theorem in the theory of Hilbert spaces (generally known as Riesz-Frechet Theorem) which states that

$$\delta\underset{\sim}{\mathfrak{R}}^{\infty}_{s=o}\{\underset{\sim}{G}_t(s);\underset{\sim}{C}\} = \int_o^\infty \underset{\sim}{\Gamma}(s;\underset{\sim}{C})\underset{\sim}{G}_t(s)ds, \qquad (4.2:13)$$

where $\underset{\sim}{\Gamma}(s;\underset{\sim}{C})$, for each s and each $\underset{\sim}{C}$, is a linear transformation of space of symmetric tensors into itself (i.e., a fourth order tensor). The constitutive relation defining finite linear viscoelasticity (4.2:12) then assumes the form

$$\hat{\underset{\sim}{T}} = \underset{\sim}{f}(\underset{\sim}{C}) + \int_o^\infty \underset{\sim}{\Gamma}(s;\underset{\sim}{C})\underset{\sim}{G}_t(s)ds. \qquad (4.2:14)$$

In the case of isotropic materials, it can be shown (see Section 3.1) that the tensor valued function $\underset{\sim}{f}(\underset{\sim}{C})$ and the linear functional given by the integral are isotropic in the sense

$$\underset{\sim}{Q}\underset{\sim}{f}(\underset{\sim}{C})\underset{\sim}{Q}^T = \underset{\sim}{f}(\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{Q}^T) \qquad (4.2:15)$$

and

$$\underset{\sim}{Q}\left[\int_0^\infty \underset{\sim}{\Gamma}(s;\underset{\sim}{C})\underset{\sim}{G}_t(s)ds\right]\underset{\sim}{Q}^T = \int_0^\infty \underset{\sim}{\Gamma}(s;\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{Q}^T)\underset{\sim}{Q}(s)\underset{\sim}{G}_t(s)\underset{\sim}{Q}(s)^T ds \ , \quad (4.2:16)$$

for all orthogonal tensors $\underset{\sim}{Q}(t)$. It has been shown (see Truesdell and Noll (15), article 37) that

$$\underset{\sim}{f}(\underset{\sim}{C}) = h_o\underset{\sim}{I} + h_1\underset{\sim}{C} + h_2\underset{\sim}{C}^2 \quad (4.2:17)$$

and

$$\begin{aligned}\underset{\sim}{\Gamma}(s;\underset{\sim}{C})\underset{\sim}{G}_t(s) &= \underset{\sim}{A}_1(s;\underset{\sim}{C})\underset{\sim}{G}_t(s) + \underset{\sim}{G}_t(s)\underset{\sim}{A}_1(s;\underset{\sim}{C}) \\ &+ \mathrm{tr}\left[\underset{\sim}{G}_t(s)\underset{\sim}{A}_2(s;\underset{\sim}{C})\right]\underset{\sim}{I} + \mathrm{tr}\left[\underset{\sim}{G}_t(s)\underset{\sim}{A}_3(s;\underset{\sim}{C})\right]\underset{\sim}{C} \\ &+ \mathrm{tr}\left[\underset{\sim}{G}_t(s)\underset{\sim}{A}_4(s;\underset{\sim}{C})\right]\underset{\sim}{C}^2 \end{aligned} \quad (4.2:18)$$

where for each s the tensor functions $\underset{\sim}{A}_i(s;\underset{\sim}{C})$ are isotropic and hence have representations of the form (4.2:17).

4.3 SMALL DEFORMATIONS - LINEAR VISCOELASTICITY

The theory of finite linear viscoelasticity is based on the assumption that the deformations are small in the recent past. However, no restrictions whatsoever, were put on deformations in remote past. In this chapter let us investigate the behavior of a simple material with fading memory for small deformations. We use the term "small deformation" in the sense as defined in Section 2.1.

Let us first consider, how the various quantities defined in terms of deformation gradients $\underset{\sim}{F}(t)$, that occur in the constitutive equation (4.2:14) of finite linear viscoelasticity, may be approximated in the case of small deformations.

From equation (2.1:22),

$$\underset{\sim}{R}^T(t) = \underset{\sim}{R}(t)^{-1} = \underset{\sim}{I} - \underset{\sim}{W}(t) + O(\epsilon^2). \qquad (4.3:1)$$

Using equations (2.1:16) and (2.1:17) in equation (2.1:25),

$$\underset{\sim}{F}_t(\tau) = \left[\underset{\sim}{I} + \underset{\sim}{H}(\tau)\right]\left[\underset{\sim}{I} + \underset{\sim}{H}(t)\right]^{-1}$$

$$= \underset{\sim}{I} + \underset{\sim}{H}(\tau) - \underset{\sim}{H}(t) + O(\epsilon^2). \qquad (4.3:2)$$

Hence $\underset{\sim}{C}_t(\tau)$ can be written as

$$\underset{\sim}{C}_t(\tau) = \underset{\sim}{F}_t(\tau)^T\underset{\sim}{F}_t(\tau)$$

$$= \underset{\sim}{I} + \left[\underset{\sim}{H}(\tau) + \underset{\sim}{H}(\tau)^T\right] - \left[\underset{\sim}{H}(t) + \underset{\sim}{H}(t)^T\right] + O(\epsilon^2)$$

$$= \underset{\sim}{I} + 2\left[\underset{\sim}{E}(\tau) - \underset{\sim}{E}(t)\right] + O(\epsilon^2)$$

$$= \underset{\sim}{I} + O(\epsilon^2). \qquad (4.3:3)$$

From the definition of $\underset{\sim}{G}_t(s)$,

$$\underset{\sim}{G}_t(s) = \underset{\sim}{R}(t)^T\underset{\sim}{C}_t(t-s)\underset{\sim}{R}(t) - \underset{\sim}{I}$$

$$= \underset{\sim}{I} + 2\left[\underset{\sim}{E}(t-s) - \underset{\sim}{E}(t)\right] + O(\epsilon^2). \qquad (4.3:4)$$

Using equation (2.1:20),

$$\underset{\sim}{f}(\underset{\sim}{C}(t)) = \underset{\sim}{f}\left[\underset{\sim}{I} + 2\underset{\sim}{E}(t) + O(\epsilon^2)\right]. \qquad (4.3:5)$$

If the material has always been held in the reference configuration, we have $\underset{\sim}{C}(t) = \underset{\sim}{I}$ and the corresponding stress $\underset{\sim}{T}_o$, called the "residual stress", is given by

$$\underset{\sim}{T}_0 = \underset{\sim}{f}(\underset{\sim}{I}) \qquad (4.3:6)$$

By using the assumption that $\underset{\sim}{f}(\underset{\sim}{C}(t))$ is continuously differentiable, we can expand $\underset{\sim}{f}(\underset{\sim}{C}(t))$ in a power series about $\underset{\sim}{I}$ and write

$$\underset{\sim}{f}(\underset{\sim}{C}(t)) = \underset{\sim}{T}_0 + \underset{\sim}{L}(\underset{\sim}{E}(t)) + O(\epsilon^2) \qquad (4.3:7)$$

where $\underset{\sim}{L}$ is a tensor valued linear function of $\underset{\sim}{E}$.

Making use of equations (2.1:22), (4.3:4) and (4.3:7) in equation (4.2:14) and neglecting terms of $O(\epsilon^n)$ for $n > 1$, we get

$$\underset{\sim}{T}(t) - \underset{\sim}{T}_0 = \underset{\sim}{W}(t)\underset{\sim}{T}_0 - \underset{\sim}{T}_0\underset{\sim}{W}(t) + \underset{\sim}{L}(\underset{\sim}{E}(t)) + 2\int_0^\infty \underset{\sim}{\Gamma}(s;\underset{\sim}{I})\left[\underset{\sim}{E}(t-s) - \underset{\sim}{E}(t)\right]ds \; . \qquad (4.3:8)$$

Let us define a new fourth order tensor $\underset{\sim}{J}(s)$ called stress relaxation function by

$$\underset{\sim}{J}(s) = - 2\int_s^\infty \underset{\sim}{\Gamma}(\xi;\underset{\sim}{I})d\xi \; . \qquad (4.3:9)$$

Then

$$\dot{\underset{\sim}{J}}(s) = \frac{d}{ds}\underset{\sim}{J}(s) = 2\underset{\sim}{\Gamma}(s;\underset{\sim}{I}) \qquad (4.3:10)$$

and hence we may write equation (4.3:8) in the form

$$\underset{\sim}{T}(t) - \underset{\sim}{T}_0 = \underset{\sim}{W}(t)\underset{\sim}{T}_0 - \underset{\sim}{T}_0\underset{\sim}{W}(t) + \left[\underset{\sim}{L} + \underset{\sim}{J}(0)\right]\underset{\sim}{E}(t) + \int_0^\infty \dot{\underset{\sim}{J}}(s)\left[\underset{\sim}{E}(t-s)\right]ds \; . \qquad (4.3:11)$$

When the residual stress $\underset{\sim}{T}_0$ is zero, the reference configuration is called the "natural state." In this state, the above constitutive equation reduces to

$$\underset{\sim}{T}(t) = \left[\underset{\sim}{L} + \underset{\sim}{J}(0)\right]\underset{\sim}{E}(t) + \int_0^\infty \dot{\underset{\sim}{J}}(s)\left[\underset{\sim}{E}(t-s)\right]ds \; . \qquad (4.3:12)$$

This is the constitutive equation of the classical theory of linear viscoelasticity. Notice that this equation is valid for materials without any symmetry at all. If the material is isotropic then we can show that $\underset{\sim}{L}$ and $\underset{\sim}{J}(s)$ should be isotropic functions of $\underset{\sim}{E}(t)$. The constitutive equation of isotropic infinitesimal viscoelasticity may be written in the form

$$\underset{\sim}{T}(t) = \{(\lambda + \overline{\lambda}(0))\mathrm{tr}\underset{\sim}{E}(t) + \int_0^\infty \overline{\lambda}(s)\mathrm{tr}\underset{\sim}{E}(t-s)ds\}\underset{\sim}{I}$$

$$+ 2(\mu + \overline{\mu}(0))\underset{\sim}{E}(t) + 2\int_0^\infty \overline{\mu}(s)\underset{\sim}{E}(t-s)ds \qquad (4.3:13)$$

where λ and μ are material constants and $\overline{\lambda}(s)$ and $\overline{\mu}(s)$ are material functions of time. The constants λ and μ coincide with the Lamé constants of the material in equilibrium. The functions $\lambda + \overline{\lambda}(t)$ and $\mu + \overline{\mu}(t)$ may be regarded as time dependent Lamé coefficients for the stress relaxation response to a sudden deformation at time $t = 0$.

4.4 ELASTICITY

The constitutive equations of the theories of nonlinear and classical linear elasticities can be deducted from the previous theory. An elastic material is defined as a simple material for which the stress at time t depends only on the local configuration at time t. Hence the constitutive equation defining as elastic material takes the form:

$$\underset{\sim}{T}(t) = \mathcal{G}(\underset{\sim}{F}(t)) \cdot \qquad (4.4:1)$$

This can be easily seen to be the case corresponding to $\underset{\sim}{\Gamma} = 0$ for finite deformations and $\underset{\sim}{J} = 0$ for infinitesimal deformations. In this case the constitutive equations (4.2:14) and (4.3:13) reduce to

$$\underset{\sim}{T}(t) = h_0\underset{\sim}{I} + h_1\underset{\sim}{B}(t) + h_2\underset{\sim}{B}(t)^2 \qquad (4.4:2)$$

and

$$T(t) = \lambda\underset{\sim}{I}tr\underset{\sim}{E}(t) + 2\mu\underset{\sim}{E}(t)\,. \qquad (4.4:3)$$

Equations (4.4:2) and (4.4:3) are the constitutive equations of isotropic nonlinear elasticity and isotropic linear (infinitesimal) elasticity respectively.

5. APPROXIMATE PHOTOMECHANICAL CONSTITUTIVE EQUATIONS

In Sections 3.2 and 3.3 we have shown that the general mechanical and photomechanical constitutive equations of an isotropic simple material has the form of equations (3.2:12) and (3.3:13). For further discussion of photomechanics, it is necessary to assume that the electromechanical constitutive equation (3.3:13) is invertible where the functionals are isotropic.

We will show in Chapter 6 that photo methods can be effectively used to "solve" two dimensional problems of classical linear elasticity and viscoelasticity: Methods are available to measure the fringe order n(t) which is proportional to the difference in the principle values of the refraction tensor

$$\underset{\sim}{N}(t) = \underset{\sim}{K}(t)^{\frac{1}{2}}. \qquad (5.1:1)$$

For the discussion of nonlinear problems in general, however, we assume that we can experimentally determine the refraction tensor $\underset{\sim}{N}(t)$ itself. Therefore, we introduce a new functional such that

$$\underset{\sim}{B}(t) = \underset{\sim}{\mathcal{Q}}\{\underset{\sim}{N}(\tau)\}_{-\infty}^{t} \tag{5.1:2}$$

where $\underset{\sim}{\mathcal{Q}}$ is an isotropic functional. Using this equation and equation (3.2:12), we get the more convenient form of the electromechanical constitutive equation

$$\begin{aligned}\underset{\sim}{T}(t) &= \underset{\sim}{T}\{\underset{\sim}{N}(\tau)\}_{-\infty}^{t} \\ &= \underset{\sim}{T}\{\underset{\sim}{N}(t-s)\}_{0}^{\infty} \ .\end{aligned} \tag{5.1:3}$$

This equation can be valid only for isotropic materials (both mechanically and electromechanically).

Define a new tensor by

$$\underset{\sim}{N}_t(t-s) = \underset{\sim}{N}(t-s)\underset{\sim}{N}(t)^{-1} \ . \tag{5.1:4}$$

We can then write equation (5.1:3) in the form

$$\underset{\sim}{T}(t) = \underset{\sim}{T}\{\underset{\sim}{N}_t(t-s)_{0}^{\infty}; \underset{\sim}{N}(t)\} \tag{5.1:5}$$

that is the stress is a functional of the history $\underset{\sim}{N}_t(\tau)$ and a function of the present value $\underset{\sim}{N}(t)$. From the definition (5.1:4),

$$\underset{\sim}{N}_t(t) = \underset{\sim}{I} \ ; \tag{5.1:6}$$

therefore, equation (5.1:5) can be written in the form analogous to equation (4.2:7):

$$\underset{\sim}{T}(t) = \underset{\sim}{\gamma}(\underset{\sim}{N}(t)) + \underset{\sim}{\mathfrak{N}}\{\overline{\underset{\sim}{N}}_t(t-\overset{\infty}{\underset{0}{s}});\underset{\sim}{N}(t)\} \ , \qquad (5.1:7)$$

where

$$\overline{\underset{\sim}{N}}_t(\tau) = \underset{\sim}{N}_t(\tau) - \underset{\sim}{I} \qquad (5.1:7a)$$

and the functional $\underset{\sim}{\mathfrak{N}}$ is such that

$$\underset{\sim}{\mathfrak{N}}\{0;\underset{\sim}{N}(t)\} = 0. \qquad (5.1:8)$$

The function $\underset{\sim}{\gamma}$ and the functional $\underset{\sim}{\mathfrak{N}}$ must be isotropic in their arguments.

We make the following assumptions similar to those that were made in Chapter 4:

1. Principle of fading memory for electromechanical case: There exists an obliviator of order greater than $n + \frac{1}{2}$ such that the response functional $\underset{\sim}{\mathfrak{N}}\{\overline{\underset{\sim}{N}}_t(t-s);\underset{\sim}{N}(t)\}$ is defined and n times Frechet differentiable in a neighborhood of the zero history of the function space H_1 of symmetric tensors $\overline{\underset{\sim}{N}}_t(t-s)$.

2. The Frechet differentiability of the response functional $\underset{\sim}{\mathfrak{N}}$ postulated in the above principle of fading memory is uniform in the tensor parameter $\underset{\sim}{N}(t)$.

3. The tensor function $\underset{\sim}{\gamma}(N(t))$ of equation (5.1:7) is n times continuously differentiable.

These assumptions are sufficient to justify the approximation

$$\underset{\sim}{T}(t) = \underset{\sim}{\gamma}(\underset{\sim}{N}(t)) + \sum_{k=1}^{n} \frac{1}{k!} \, \delta^k \underset{\sim}{\mathfrak{N}}\{\overline{\underset{\sim}{N}}_t(t-s), \underset{\sim}{N}(t)\}$$

$$+ 0(\|\underset{\sim}{N}_t(t-s) - \underset{\sim}{I}\|_h^n) \qquad (5.1:9)$$

where the norm of $\underset{\sim}{N}$ is defined similar to that of $\underset{\sim}{G}$ (see equation 4.1:5) and the order symbol is used in the sense of equation (4.2:10). In particular,

$$\underset{\sim}{T}(t) = \underset{\sim}{\gamma}(\underset{\sim}{N}(t)) + \delta\underset{\sim}{\mathfrak{N}}\overset{\infty}{\underset{s=0}{}}\{\underset{\sim}{N}_t(t-s) - \underset{\sim}{I};\underset{\sim}{N}(t)\}$$

$$+ 0(\|\underset{\sim}{N}_t(t-s) - \underset{\sim}{I}\|_h^2) . \qquad (5.1:10)$$

For deformations such that the histories $\underset{\sim}{N}_t(t-s)$ are small in recent past, we can neglect the last term in this equation:

$$\underset{\sim}{T}(t) \simeq \underset{\sim}{\gamma}(\underset{\sim}{N}(t)) + \delta\underset{\sim}{\mathfrak{N}}\overset{\infty}{\underset{s=0}{}}\{\underset{\sim}{N}_t(t-s);\underset{\sim}{N}(t)\} . \qquad (5.1:11)$$

By the Riesz-Frechet theorem, the linear tensor valued functional $\delta\underset{\sim}{\mathfrak{N}}$ can be written as

$$\delta\underset{\sim}{\mathfrak{N}}\overset{\infty}{\underset{s=0}{}}\{\underset{\sim}{N}_t(t-s);\underset{\sim}{N}(t)\} = \int_0^\infty \underset{\sim}{\Psi}(s;\underset{\sim}{N})\underset{\sim}{N}_t(t-s)ds \qquad (5.1:12)$$

where $\underset{\sim}{\Psi}(s;\underset{\sim}{N})$ is a fourth order tensor. Thus, we can write equation (5.1:11) in the form

$$\underset{\sim}{T}(t) = \underset{\sim}{\gamma}(\underset{\sim}{N}(t)) + \int_0^\infty \underset{\sim}{\Psi}(s;\underset{\sim}{N})\underset{\sim}{N}_t(t-s)ds . \qquad (5.1:13)$$

As the functions are isotropic, they must satisfy the identities

$$\underset{\sim}{Q}(t)\underset{\sim}{\gamma}(\underset{\sim}{N}(t))\underset{\sim}{Q}(t)^T = \underset{\sim}{\gamma}(\underset{\sim}{Q}(t)\underset{\sim}{N}(t)\underset{\sim}{Q}(t)^T) \qquad (5.1:14)$$

and

$$\underset{\sim}{Q}(t)\left[\int_0^\infty \Psi(s;\underset{\sim}{N})\{\underset{\sim}{N}_t(t-s)\}ds\right]\underset{\sim}{Q}(t)^T =$$

$$\int_0^\infty \underset{\sim}{\Psi}(s;\underset{\sim}{Q}\underset{\sim}{N}\underset{\sim}{Q}^T)\underset{\sim}{Q}(t-s)\underset{\sim}{N}_t(t-s)\underset{\sim}{Q}(t-s)^T ds \qquad (5.1:15)$$

for all orthogonal tensors $\underset{\sim}{Q}$. Thus they have representations similar to equations (4.2:17) and (4.2:18), i.e.,

$$\underset{\sim}{\gamma}(\underset{\sim}{N}(t)) = \hat{h}_0\underset{\sim}{I} + \hat{h}_1\underset{\sim}{N}(t) + \hat{h}_2\underset{\sim}{N}(t)^2 \qquad (5.1:16)$$

and

$$\begin{aligned}\underset{\sim}{\Psi}(s;\underset{\sim}{N})\underset{\sim}{N}_t(t-s) &= \overline{\underset{\sim}{A}}_1(s;\underset{\sim}{N})\underset{\sim}{N}_t(t-s) + \underset{\sim}{N}_t(t-s)\overline{\underset{\sim}{A}}_1(s;\underset{\sim}{N}) \\ &+ \mathrm{tr}\left[\underset{\sim}{N}_t(t-s)\overline{\underset{\sim}{A}}_2(s;\underset{\sim}{N})\right]\underset{\sim}{I} \\ &+ \mathrm{tr}\left[\underset{\sim}{N}_t(t-s)\overline{\underset{\sim}{A}}_3(s;\underset{\sim}{N})\right]\underset{\sim}{N}(t) \\ &+ \mathrm{tr}\left[\underset{\sim}{N}_t(t-s)\overline{\underset{\sim}{A}}_4(s;\underset{\sim}{N})\right]\underset{\sim}{N}(t)^2 \qquad (5.1:17)\end{aligned}$$

where, for each s, the tensors $\overline{\underset{\sim}{A}}_i(s;N)$ are isotropic and hence have representations of the form (5.1:16).

In order to further simplify equation (5.1:13), we consider the following approximation. Define a new tensor $\underset{\sim}{N}'(t)$ as follows:

$$\underset{\sim}{N}'(\tau) = \underset{\sim}{N}^{-1}(0)\underset{\sim}{N}(\tau) - \underset{\sim}{I} \; . \qquad (5.1:18)$$

Then,

$$N'(0) = 0 \; . \qquad (5.1:19)$$

We wish to consider the case when N' remains small. Let

$$\varepsilon = \sup_{\tau} \left| \underset{\sim}{N}'(\tau) \right| \; , \qquad (5.1:20)$$

where $\left|N'(\tau)\right|$ is given by (4.1:6). We say that the changes in refraction tensor are small if

$$\epsilon << 1 \qquad (5.1:21)$$

and we consider deformations such that the above inequality is true.

By the third assumption made in the beginning of this chapter, we can expand $\underset{\sim}{\gamma}$ in a power series of $\underset{\sim}{N}'(t)$:

$$\underset{\sim}{\gamma}(\underset{\sim}{N}(t)) = \underset{\sim}{\gamma}(\underset{\sim}{N}(0)) + \underset{\sim\sim}{L}\underset{\sim}{N}'(t) + 0(\epsilon^2) \qquad (5.1:22)$$

where $\underset{\sim}{L}$ is a fourth order tensor. Because of the assumption that the stress is zero in the reference state,

$$\underset{\sim}{\gamma}(\underset{\sim}{N}(0)) = 0 \ . \qquad (5.1:23)$$

Thus,

$$\underset{\sim}{\gamma}(\underset{\sim}{N}(t)) = \underset{\sim\sim}{L}\underset{\sim}{N}'(t) + 0(\epsilon^2) \ . \qquad (5.1:24)$$

Furthermore, we get

$$\underset{\sim}{N}_t(\tau) = \underset{\sim}{I} + \underset{\sim}{N}'(\tau) - \underset{\sim}{N}'(t) + 0(\epsilon^2) \ . \qquad (5.1:25)$$

Making use of equations (5.1:24) and (5.1:25) in equation (5.1:13) and neglecting all terms of $0(\epsilon^n)$ for $n > 1$, we get,

$$\underset{\sim}{T}(t) = \underset{\sim\sim}{L}\underset{\sim}{N}'(t) + \int_0^\infty \underset{\sim}{\Psi}(s;\underset{\sim}{I})\left[\underset{\sim}{N}'(t-s)-\underset{\sim}{N}'(t)\right]ds \ . \qquad (5.1:26)$$

Let us define a new function $\underset{\sim}{\Phi}(s)$ called optical relaxation function by

$$\underset{\sim}{\Phi}(s) = -\int_0^\infty \underset{\sim}{\Psi}(\xi,\underset{\sim}{I})d\xi \ . \qquad (5.1:27)$$

Then

$$\underset{\sim}{\Psi}(s;\underset{\sim}{I}) = \frac{d}{ds}\underset{\sim}{\Phi}(s) = \dot{\underset{\sim}{\Phi}}(s) \ . \qquad (5.1:28)$$

Using equation (5.1:28), we can write equation (5.1:26) in the form

$$\underset{\sim}{T}(t) = \left[\underset{\sim}{L} + \underset{\sim}{\Phi}(0)\right]\underset{\sim}{N}'(t) + \int_0^\infty \dot{\underset{\sim}{\Phi}}(s)\left[\underset{\sim}{N}'(t-s)\right]ds \ . \qquad (5.1:29)$$

The functions $\underset{\sim}{L}$ and $\underset{\sim}{\Phi}$ must be isotropic functions of $\underset{\sim}{N}(t)$ and thus can be represented in the form of equation (5.1:16). Using this representation, and simplifying the above isotropic electromechanical constitutive equation for the case of infinitesimal changes in the dielectric tensor, we obtain

$$\underset{\sim}{T}(t) = \left[(\alpha + \overline{\alpha}(0))\mathrm{tr}\underset{\sim}{N}'(t) + \int_0^\infty \dot{\overline{\alpha}}(s)\mathrm{tr}\underset{\sim}{N}'(t-s)ds\right]\underset{\sim}{I}$$

$$+ 2(\beta + \overline{\beta}(0))\underset{\sim}{N}'(t) + 2\int_0^\infty \dot{\overline{\beta}}(s)\underset{\sim}{N}'(t-s)ds \qquad (5.1:30)$$

where α and β are material constants and $\alpha(t)$ and $\beta(t)$ are material functions which must be evaluated experimentally.

So far, we have discussed time dependent photomechanical equations. But for many materials, such as those used in photoelastic experiments, the stress at any time t can be determined by the value of $\underset{\sim}{N}(\tau)$ at time t. This corresponds to the case where $\Psi = 0$. In such cases, the time dependent material functions vanish in the photomechanical constitutive equations and equations (5.1:13) and (5.1:30) simplify to

$$\underset{\sim}{T}(t) = \hat{h}\underset{\sim}{I} + \hat{h}_1\underset{\sim}{N}'(t) + \hat{h}_2\underset{\sim}{N}'(t)^2 \qquad (5.1:31)$$

for the finite linear case, and

$$\underset{\sim}{T}(t) = \alpha \underset{\sim}{I} \mathrm{tr} \underset{\sim}{N}'(t) + 2\beta \underset{\sim}{N}'(t) \qquad (5.1{:}32)$$

for the case of infinitesimal changes in the dielectric tensor.

6. LINEAR PHOTOMECHANICS

6.1 LINEAR PHOTOVISCOELASTICITY

By linear photoviscoelastcity we mean the experimental determination of stresses by photo methods in bodies governed by the constitutive equation of isotropic infinitesimal viscoelastcity. This theory was explained by Dill in a series of articles (References [4], [5], and [6]). It is an important special case of the general theory as given in this article. The presentation of this section follows those articles closely.

The constitutive relation of isotropic, infinitesimal viscoelasticity is given by equation (4.3:13). Changing the variable from s to $\tau = t-s$, equation (4.3:13) becomes

$$\underset{\sim}{T}(t) = \left\{(\lambda+\bar{\lambda}(0))\mathrm{tr}\underset{\sim}{E}(t) + \int_{-\infty}^{t} \dot{\bar{\lambda}}(t-\tau)\mathrm{tr}\underset{\sim}{E}(\tau)d\tau\right\}\underset{\sim}{I}$$

$$+ 2(\mu+\bar{\mu}(0))\underset{\sim}{E}(t) + 2\int_{-\infty}^{t} \dot{\bar{\mu}}(t-\tau)\underset{\sim}{E}(\tau)d\tau \ . \qquad (6.1{:}1)$$

It is assumed that the material is in its natural configuration up to time $\tau = 0$. In such a case, the lower limits in the integrals of equation (6.1:1) can be replaced by zeros. Furthermore, introducing new material functions defined by

$$\lambda_1(t) \equiv \lambda+\bar{\lambda}(t) \qquad (6.1{:}2)$$

and

$$\mu_1(t) \equiv 2(\mu+\bar{\mu}(t)) \ ; \tag{6.1:3}$$

equation (6.1:1) may be written as

$$\underset{\sim}{T}(t) = \left[\lambda_1(t)\mathrm{tr}\underset{\sim}{E}(t) + \int_0^t \dot{\lambda}_1(t-\tau)\mathrm{tr}\underset{\sim}{E}(\tau)d\tau\right]\underset{\sim}{I}$$

$$+ \mu_1(t)\underset{\sim}{E}(t) + \int_0^t \dot{\mu}_1(t-\tau)\underset{\sim}{E}(\tau)d\tau \ . \tag{6.1:4}$$

Integration by parts reduces this equation to

$$\underset{\sim}{T}(t) = \int_0^t\left[\underset{\sim}{I}\lambda_1(t-\tau)\mathrm{tr}\dot{\underset{\sim}{E}}(\tau) + \mu_1(t-\tau)\dot{\underset{\sim}{E}}(\tau)\right]d\tau \ . \tag{6.1:5}$$

Taking the trace of both sides yields

$$\mathrm{tr}\underset{\sim}{T}(t) = \int_0^t\left[3\lambda_1(t-\tau) + \mu_1(t-\tau)\right]\mathrm{tr}\dot{\underset{\sim}{E}}(\tau)d\tau \ . \tag{6.1:6}$$

It is usual to write this in the form

$$\tfrac{1}{3}\mathrm{tr}\underset{\sim}{T}(t) = \int_0^t K(t-\tau)\mathrm{tr}\underset{\sim}{E}(\tau)d\tau \ , \tag{6.1:7}$$

where

$$K(\xi) = \lambda_1(\xi) + \tfrac{1}{3}\mu_1(\xi) \tag{6.1:8}$$

is called the bulk relaxation modulus.

From equations (6.1:5) and (6.1:6),

$$\underset{\sim}{S}(t) = \int_0^t \mu_1(t-\tau)\dot{\underset{\sim}{\varepsilon}}(\tau)d\tau \ ; \tag{6.1:9}$$

where

$$\underset{\sim}{S}(t) = \underset{\sim}{T}(t) - \tfrac{1}{3}\mathrm{tr}\underset{\sim}{T}(t)\underset{\sim}{I} \tag{6.1:10}$$

and

$$\underset{\sim}{\varepsilon}(t) = \underset{\sim}{E}(t) - \tfrac{1}{3}\mathrm{tr}\underset{\sim}{E}(t)\underset{\sim}{I} \tag{6.1:11}$$

are called stress deviator and strain deviator respectively. The material function μ_1 is generally called shear relaxation modulus.

So far, we have discussed only the mechanical constitutive equation; now we consider the photomechanical constitutive equation. We call a material linear photoviscoelastic if its mechanical constitutive equation is of the form (6.1:1) and its electromechanical constitutive equation is of the form of equation (5.1:30). With the change of variable $\tau = t-s$, equation (5.1:30) takes the form

$$\underset{\sim}{T}(t) = \left[(\alpha+\bar{\alpha}(0))\,\mathrm{tr}\underset{\sim}{N}'(t) + \int_{-\infty}^{t} \dot{\alpha}(t-\tau)\,\mathrm{tr}\underset{\sim}{N}'(\tau)\,d\tau \right]\underset{\sim}{I}$$

$$+ 2(\beta+\bar{\beta}(0))\underset{\sim}{N}'(t) + \int_{-\infty}^{t} \dot{\beta}(t-\tau)\underset{\sim}{N}'(\tau)\,d\tau \,. \qquad (6.1:12)$$

If we assume that the body is in its natural configuration for all times $\tau < 0$, we can replace the lower limits of the integrals in the above equation by zeros:

$$\underset{\sim}{T}(t) = \left\{ (\alpha+\bar{\alpha}(0))\,\mathrm{tr}\underset{\sim}{N}'(t) + \int_{0}^{t} \dot{\alpha}(t-\tau)\,\mathrm{tr}\underset{\sim}{N}'(\tau)\,d\tau \right\}\underset{\sim}{I}$$

$$+ 2(\beta+\bar{\beta}(0))N'(t) + \int_{0}^{t} \dot{\beta}(t-\tau)N'(\tau)\,d\tau \,. \qquad (6.1:13)$$

Let us define new material functions $\alpha_1(t)$ and $\beta_1(t)$ as follows:

$$\alpha_1(t) \equiv \alpha+\bar{\alpha}(t) \qquad (6.1:14)$$

and

$$\beta_1(t) \equiv \beta+\bar{\beta}(t) \,. \qquad (6.1:15)$$

Introducing these new material functions in equation (6.1:13)

and using integration by parts, we obtain

$$\underset{\sim}{T}(t) = \int_0^t \left[\underset{\sim}{I}\alpha_1(t-\tau)\,\mathrm{tr}\dot{\underset{\sim}{N}}'(\tau) + \beta_1(t-\tau)\dot{\underset{\sim}{N}}'(\tau) \right] d\tau \ . \qquad (6.1{:}16)$$

Taking the trace of the above equation, we get

$$\mathrm{tr}\underset{\sim}{T}(t) = \int_0^t \left[3\alpha_1(t-\tau) + \beta_1(t-\tau) \right] \mathrm{tr}\dot{\underset{\sim}{N}}'(\tau)\, d\tau \ . \qquad (6.1{:}17)$$

for an isotropic material $\underset{\sim}{N}(0) = N_0\underset{\sim}{I}$. Then

$$N_0\dot{\underset{\sim}{N}}'(\tau) = \dot{\underset{\sim}{N}}(\tau) \ .$$

The equation can be written in the form

$$\mathrm{tr}\underset{\sim}{T}(t) = \int_0^t \varphi(t-\tau)\,\mathrm{tr}\dot{\underset{\sim}{N}}(\tau)\, d\tau \ , \qquad (6.1{:}18)$$

where

$$\varphi(\xi) = \left[3\alpha_1(\xi) + \beta_1(\xi) \right] \Big/ N_0 \qquad (6.1{:}19)$$

is called the optical bulk relaxation function.

From equations (6.1:16) and (6.1:17), we can write

$$\underset{\sim}{S}(t) = \int_0^t \psi(t-\tau)\dot{\underset{\sim}{P}}(\tau)\, d\tau \ ; \qquad (6.1{:}20)$$

where

$$\underset{\sim}{P}(t) = \underset{\sim}{N}(t) - \frac{1}{3}\,\mathrm{tr}\underset{\sim}{N}(t)\underset{\sim}{I} \qquad (6.1{:}21)$$

and

$$\psi(\xi) = \frac{1}{N_0}\mu_1(\xi) \ . \qquad (6.1{:}22)$$

$\underset{\sim}{P}$ is deviator of the refraction tensor and ψ is the optical relaxation function

Equation (6.1:20) can be used to calculate the difference in principal stresses for a two dimensional linear viscoelastic

problem, by using experimentally measured data. The reader is referred to the articles by Dill and Fowlkes (References [6] and [7]) for a detailed description of the experimental procedure.

6.2 LINEAR PHOTOELASTICITY

In Section 4.4 we defined an elastic material as a simple material in which the stress at time t depends only on the deformation gradient at time t. In the mechanical constitutive equation of such a material, all time dependent material functions vanish, i.e., $\lambda(t)$ and $\mu(t)$ are identically zero. This case is represented by equation (4.4:3). In terms of deviatoric quantities we can write

$$\underset{\sim}{S} = 2\mu\underset{\sim}{\varepsilon} \tag{6.2:1}$$

and

$$\frac{1}{3}\mathrm{tr}\underset{\sim}{T} = (\lambda+\frac{2}{3}\mu)\mathrm{tr}\underset{\sim}{E} \tag{6.2:2}$$

A material for which the stress tensor at any time t depends only on the refraction tensor at that time will be called "photoelastic". Their photomechanical constitutive equation can be written as

$$\underset{\sim}{T} = \underset{\sim}{I}\alpha\mathrm{tr}\underset{\sim}{N}' + 2\beta\underset{\sim}{N}' \ . \tag{6.2:3}$$

Taking the trace of this equation

$$\begin{aligned}\mathrm{tr}\underset{\sim}{T} &= 3\alpha\mathrm{tr}\underset{\sim}{N}' + 2\beta\mathrm{tr}\underset{\sim}{N}' \\ &= (3\alpha + 2\beta)\mathrm{tr}\underset{\sim}{N}'\end{aligned} \tag{6.2:4}$$

From equation (6.2:3) and (6.2:4), we obtain

$$\underset{\sim}{S} = \frac{2\beta}{N_0}\underset{\sim}{P} \qquad (6.2:5)$$

and

$$\frac{1}{3}\,\mathrm{tr}\underset{\sim}{T} = \frac{1}{N_0}(\alpha + \frac{2}{3}\beta)\mathrm{tr}\underset{\sim}{N}\ . \qquad (6.2:6)$$

These two equations are the basic equations for the theory of classical linear photoelasticity.

7. NONLINEAR PHOTOMECHANICS

7.1 NONLINEAR PHOTOVISCOELASTCITY

We say that a material is a photoviscoelastic material if its mechanical constitutive equation is given by equation (3.2:12) and if its electromechanical constitutive equation is given by equation (3.3:13) and both functionals have fading memory. For the discussion of nonlinear photomethods, we will assume that it is possible to experimentally determine the tensor $\underset{\sim}{N}(t)$. Thus, we consider $\underset{\sim}{N}(t)$ to be a known function of time.

Suppose we have a problem of stress analysis of a structure made of viscoelastic material. The main difficulty in solving this problem by photomethods is finding a suitable finite linear photoviscoelastic material to serve as model material which is transparent and with mechanical properties the same as the prototype. When such a material is found, a scale model of the actual structure is made. This model is subjected to loads proportional to those that are acting on the actual structure. By means of the appropriate optical instrument, the history of the refraction tensor $\underset{\sim}{N}(t)$ as a function to time is measured at the point where the

stress tensor is to be evaluated. Knowing the history of $\underset{\sim}{N}(t)$ and other material functions, the Cauchy Stress tensor in the model structure can be found from equation (5.1:3)

$$\underset{\sim}{T}(t) = \underset{\sim}{\mathcal{F}}\{\underset{\sim}{N}(\overset{t}{\underset{0}{\tau}})\} \ . \qquad (7.1:1)$$

The stress in the actual structure is then evaluated by multiplying the above stress tensor by the appropriate scale factor.

7.2 NONLINEAR PHOTOELASTICITY

In this Section we consider the theory of nonlinear photoelasticity by which we mean the determination of stresses by photomethods in nonlinear elastic materials. We define a nonlinear photoelastic material to be a transparent dielectric whose mechanical constitutive equation is of the form of equation (4.4:2) and whose photomechanical constitutive equation is of the form of equation (5.1:31).

The stress distribution in structures made of nonlinear elastic materials can be determined experimentally by photomethods if we can find a suitable nonlinear photoelastic model material. The requirement is that the mechanical constitutive equations of the model and actual materials are the same. By performing experiments similar to those explained in the previous section, the stress tensor in the structure can be determined.

Since every viscoelastic material is an elastic material for sufficiently slow loading, the model material may be viscoelastic. If, for example, the deformations are held constant for recent

time, then, by equation (4.2:9), we have

$$\hat{\underset{\sim}{T}} = \underset{\sim}{f}(\underset{\sim}{C}) \ . \qquad (7.2:1)$$

This is the constitutive relation of a nonlinearly elastic material. From equation (5.1:9) we have, in this case,

$$\underset{\sim}{T}(t) = \underset{\sim}{\gamma}(\underset{\sim}{N}(t)) \qquad (7.2:2)$$

for an isotropic material. In particular, this means that if the model is allowed to reach an equilibrium state under constant load, the stress state is that of an elastic material with constitutive equation (7.2:1).

7.3 PROPORTIONAL LOADING

We consider a problem for which the stress at each point experience a step change with time; that is, the stress tensor is zero for $t < 0$. In this case, we say that the structure experiences proportional loading. We will now show that the stress state is the solution to a family of nonlinear elasticity problems.

It was shown in Section (3.2) that the mechanical constitutive equation of an isotropic simple material is

$$\underset{\sim}{T} = \mathfrak{I}\left\{\underset{\sim}{B}(\overset{t}{\underset{-\infty}{\tau}})\right\} \ .$$

We will consider the class of problems for which this constitutive equation is invertible and can be written in the form

$$\underset{\sim}{B}(t) = \mathfrak{B}\left\{\underset{\sim}{T}(\overset{t}{\underset{-\infty}{\tau}})\right\} \ . \qquad (7.3:1)$$

Consider the process of creep in which the particle experiences a stress history which is zero for $\tau \leq 0$ and constant for $\tau > 0$, i.e.,

$$\underset{\sim}{T}(\tau) = \underset{\sim}{T}_0 h(\tau) \tag{7.3:2}$$

where $h(\tau)$ is the unit step function defined by

$$h(\tau) \quad \begin{cases} 0 \text{ for } \tau \leq 0 \\ 1 \text{ for } \tau > 0 \cdot \end{cases} \tag{7.3:3}$$

For such a stress history, equation (7.3:1) takes the form

$$\begin{aligned} \underset{\sim}{B}(t) &= \underset{\sim}{\mathfrak{B}}\left\{\underset{\sim}{T}_0 h(\overset{t}{\underset{-\infty}{\tau}})\right\} \\ &= \overline{\underset{\sim}{B}}(\underset{\sim}{T}_0, t) \cdot \end{aligned} \tag{7.3:4}$$

For each particular value of t, equation (7.3:4) is identical to the mechanical constitutive equation of a nonlinear elastic material. Thus, equation (7.3:4) can be considered as a one parameter family of mechanical constitutive equations, the parameter being the continuous variable t. Different values of the parameter t define different nonlinear elastic materials. Suppose we have solved the problem of stress analysis within a viscoelastic structure and find the stress is constant. This problem could have been solved as shown previously by photomethods. The stress field in such a structure at any time $t = t_1$ coincides with the stress field in a similar structure made of a nonlinear elastic material whose mechanical constitutive equation is given by

$$\underset{\sim}{B} = \overline{\underset{\sim}{B}}(\underset{\sim}{T}_0, t_1) \cdot \tag{7.3:5}$$

Thus, by knowing the solution of the viscoelastic problem, we also know the stress distribution in similar structures made of one of a family of nonlinear elastic materials.

We conclude this section by considering an example illustrating the procedure. Consider a bar of finite linear viscoelastic material under uniform axial loading. Let the mechanical constitutive equation of the material under the uniaxial loading be given by

$$T(t) = G(0)F(\lambda(t)) + \int_{-\infty}^{t} \dot{G}(t-\tau)F(\lambda(t))d\tau \qquad (7.3:6)$$

where $G(t)$ is a material function, $\lambda(t)$ is a time dependent stretch ratio and $F(\lambda(t))$ is a function of $\lambda(t)$. This equation is given by Staverman and Schwarzl [19] on the basis of experimental results. It has been shown by Lianis [18] to be a special case of equation (4.2:14). If we assume the material to be in its natural state for $t < 0$, the lower limit of the integral can be replaced by zero. The equation can be inverted:

$$f(t) = J(0)T(t) + \int_{0}^{t} \dot{J}(t-\tau)T(\tau)d\tau \ , \qquad (7.3:7)$$

where $J(t)$ is defined by the relation

$$\int_{0}^{t} G(t-\tau)J(\tau)d\tau = t \qquad (7.3:8)$$

and

$$f(t) = F(\lambda(t)) \ . \qquad (7.3:9)$$

In a creep process, the applied stress field is given by equation (7.3:2), and equation (7.3:9) yields

$$F(\lambda(t)) = T_0 J(t) \cdot \qquad (7.3:10)$$

At any particular time t_0, we get

$$F(\lambda(t_0)) = T_0 J(t_0) \cdot \qquad (7.3:11)$$

Different functions F give different equations, each of which define a certain nonlinear elastic material.

It should be clear that one cannot solve all elasticity problems this way. Only those problems may be treated in which the stress state is the same for both a viscoelastic and an elastic material. This implies that the stress state is independent, in some respect, of the material properties.

7.4 REMARKS ON PHOTOPLASTICITY

In recent literature on photomethods of stress analysis, there were attempts to extend the methods of classical photoelasticity to the case of materials in elastic-plastic states (References 10, 11, 12). These methods are generally known under the name photoplasticity. In this section, we will consider the connection between such methods and the general theory of photomechanics.

Let us consider the uniaxial deformation of two bodies made of different materials, one a nonlinear elastic material and the other a work hardening elastic-plastic material. Suppose the applied load is monotonically increasing. Under such a load, the stress-strain curve for both materials looks identical and one cannot distinguish an elastic-plastic material from a nonlinear

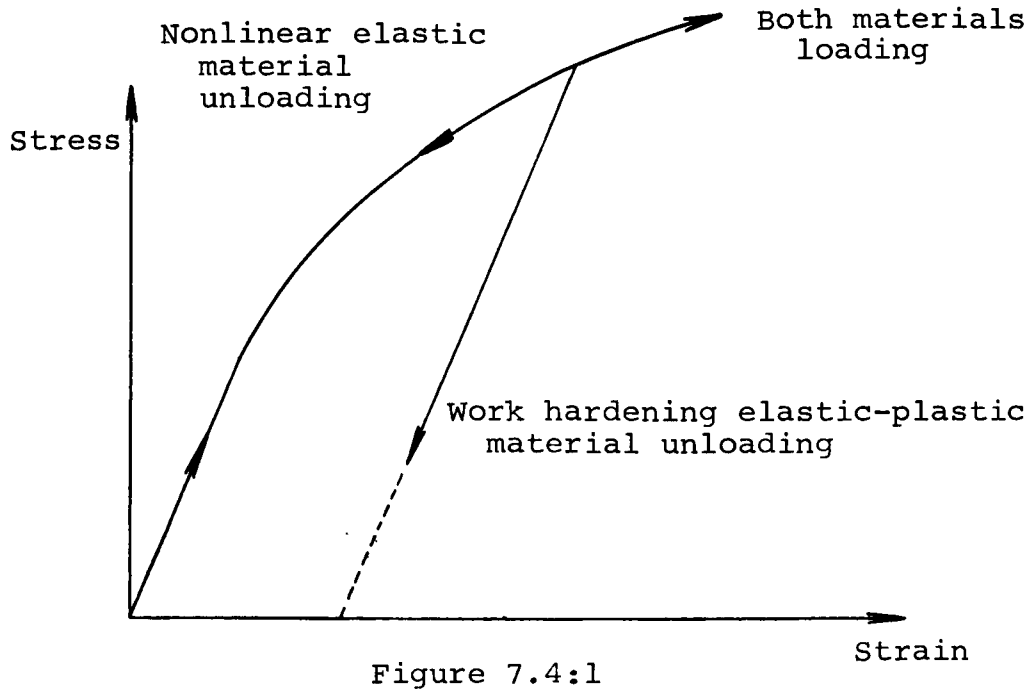

Figure 7.4:1

elastic material. The real characteristics of these materials appear only when there is unloading. During unloading, the nonlinear elastic material retraces the curve backward, whereas the elastic-plastic material follows a straight line parallel to the initial slope of the stress-strain curve (see Figure 7.4:1). A similar result is true in the case of two and three dimensional problems. Thus, under loads where no point experiences unloading, the stress analysis problems for work hardening plastic materials are indistinguishable from those for nonlinear elastic materials. The plasticity problems considered in Reference [10] and [11] seem to involve no unloading. Our remarks on nonlinear elasticity therefore apply to their tests. Furthermore, the authors appear to assume that the stress is independent of time. If their

assumption is correct, then Section 8.2 applies; if not, their interpretation of their experiments is questionable.

If the problem involves an elastic-plastic material and some point experiences unloading during the deformation process, then the stress state cannot in general be determined by observations on a viscoelastic model. It would be necessary to discover a model material which is a transparent dielectric and which, at least in the limit of slow loading, behaves as an elastic-plastic material. A theory of this kind is presented in Reference [12]. We know of no such material.

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